Homework for November 18, 2018.

Algebra.

Review the previous classwork handout. Solve the remaining problems from the previous homework assignments and classwork exercises. Try solving the following problems.

1. Using the method of mathematical induction, prove the following equality,

$$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1$$

2. Put the sign $<$, $>$, or $=$, in place of ... below,

$$\frac{n+1}{2} \ldots \sqrt[n]{n!}$$

3. Find the following sum.

$$\left(2+\frac{1}{2}\right)^2 + \left(4+\frac{1}{4}\right)^2 + \cdots + \left(2^n+\frac{1}{2^n}\right)^2$$

4. The lengths of the sides of a triangle are three consecutive terms of the geometric series. Is the common ratio of this series, q, larger or smaller than 2?
5. Solve the following equation,

$\frac{x-1}{x} + \frac{x-2}{x} + \frac{x-3}{x} + \cdots + \frac{1}{x} = 3$, where x is a positive integer.

6. Find the following sum,
   a. $1 + 2 \cdot 3 + 3 \cdot 7 + \cdots + n \cdot (2^n - 1)$
   b. $1 \cdot 3 + 3 \cdot 9 + 5 \cdot 27 + \cdots + (2n-1) \cdot 3^n$
7. Numbers $a_1, a_2, \ldots, a_n$ are the consecutive terms of a geometric progression, and the sum of its first n terms is S_n. Show that,

$$S_n = a_1 a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

8. Prove that three terms shown below are the three terms of the geometric progression, and find the sum of its first n terms, beginning with the first one below,

$$\frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{1}{3-\sqrt{3}}+\frac{1}{6}+\cdots$$

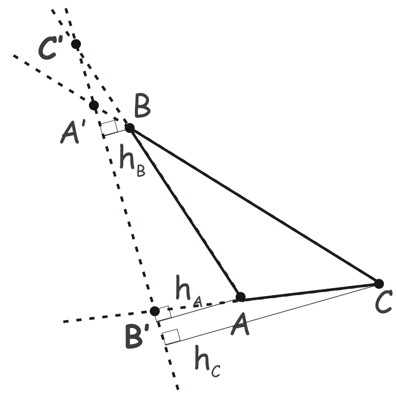
9. What is the maximum value of the expression, $(1+x)^{36}+(1-x)^{36}$ in the interval $|x|\leq 1$?

10. Find the coefficient multiplying x^9 after all parenthesis are expanded in the expression, $(1+x)^9+(1+x)^{10}+\cdots+(1+x)^{19}$.

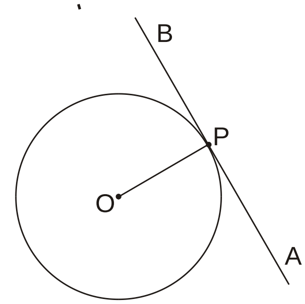
Geometry.

Review the last classwork handout on inscribed angles and quadrilaterals. Go over the proof of Ptolemy's theorem. Solve the unsolved problems from previous homework. Try solving the following problems.

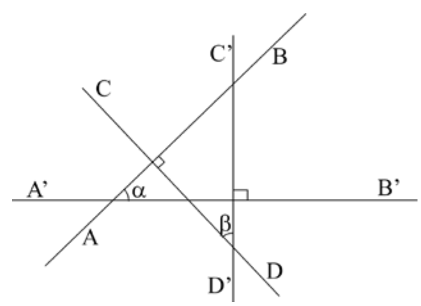
Problems.



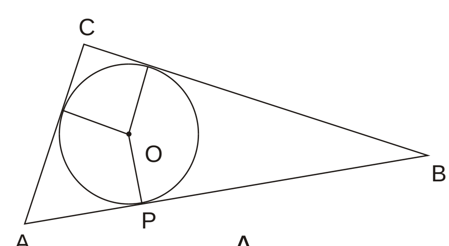
1. Prove Menelaus theorem for the configuration shown on the right using mass points. Menelaus theorem states,
   Points C', A' and B', which belong to the lines containing the sides AB, BC and CA, respectively, of triangle ABC are collinear if and only if,
   $\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|} = 1$



2. **Tangent line** to a circle is a line that has one and only one common point with the circle (definition). Prove that tangent line AB is perpendicular to the radius OP ending at the point P, which is the common point of the line and the circle (see Figure on the right).
3. We know from geometry that a circle can be drawn through the three vertices of any triangle. Find a radius of such circle if the sides of the triangle are 6, 8, and 10. (Gelfand and Saul "Trigonometry" p60, #4).

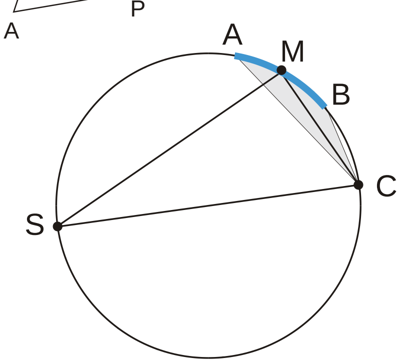


4. Prove that in the Figure on the right, $\angle\alpha$ is congruent to $\angle\beta$ if $AB \perp CD$ and $A'B' \perp C'D'$.
5. Using a compass and a ruler, draw a circle inscribed in the given triangle ABC. Prove the following formula for the area of the triangle,



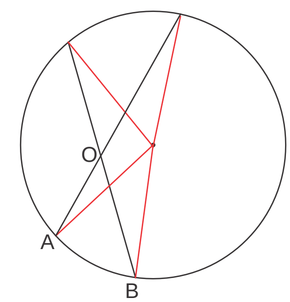
$$S_{ABC} = ½pr,$$

where p is the perimeter of the triangle and r the radius of the inscribed circle.



6. A **Rowland focusing** mirror is a device which focuses light of a certain color from the point source S onto a

point, C, at sample. The mirror has the shape of a circular arc AB of 40 cm length. It is positioned so that its center, M, is at a distance of 4 m from the source S and at a distance 2 m from the sample C, $|SM| = 4$ m, $|MC| = 2$ m. The light ray of the color of interest is reflected so that it forms a 90˚ angle with the incident ray (e.g. angle SMC in the figure on the right is 90˚).



a. What is the radius of the Rowland circle?
b. What is the angular size of the light beam illuminating the sample (shaded angle ACB in the figure)? Does it depend on the position of sample, C?

7. Prove that an angle whose vertex lies inside a disk is measured by a semi-sum of the two arcs, one of which is intercepted by this angle, and the other by the angle vertical to it.
8. Prove that an angle whose vertex lies outside a disk and whose sides intersect the circle, is measured by a semi-difference the two intercepted arcs.