November 11, 2018

Geometry.

The Inscribed Angle Theorem.

Theorem. An angle α inscribed in a circle is half of the central angle $\beta = 2\alpha$ that subtends the same arc on the circle (Fig.1), or complete half of it to 180.
Corollary. The angle does not change as its apex is moved to different positions on the circle.

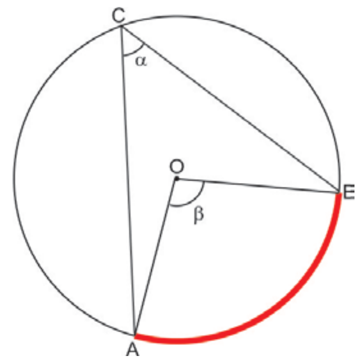


Fig. 1

Proof. First, let us deal with the simple case when one of the rays of angle ACB' passes through the center of the circle (Fig. 2). $\angle AOB'(\beta)$ is a central angle that subtends the same arc as $\angle ACB'$ (α). Triangle AOC is an isosceles triangle because $|OA| = |OC|$, so angle $\angle OAC$ and angle $\angle OCA$ are equal and angle $\angle AOC = 180\ -\ 2\alpha$, but it is also equal $180 - \beta$ as a supplement angel to angle β.

$$\angle AOC = 180 - 2\alpha\ = 180 - \beta\ \Rightarrow \beta\ =\ 2\alpha.$$

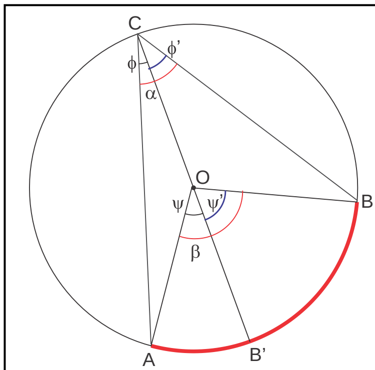


Fig. 2

In the case when center of the circle placed inside of angle ACB we can divide the angle ACB with a ray CB' passing through the center of the circle (Fig. 3). Now we have two inscribed angles: angle ACB' and angle $B'CB$, each of them has one side which passes through the center of the circle and can use previous part to proof that $\beta = 2\alpha$.

$$\alpha\ =\ \phi\ +\ \phi',$$

$$\beta\ =\ \psi\ +\ \psi'\ =\ 2\phi\ +\ 2\phi'\ =\ 2(\phi\ \ +\ \phi')\ =\ 2\alpha.$$

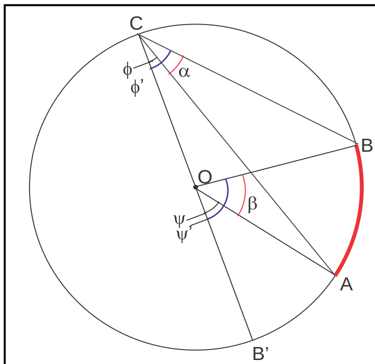


Fig. 3

When center of the circle is outside of inscribed angle, we can draw a ray from a vertex of our angle through the center the circle (Fig. 4). Then the angle $\angle ACB(\alpha) = \angle B'CB(\phi') - \angle B'CA(\phi)$ and we again can use the first part.

$$\beta = \psi' - \psi = 2\phi' - 2\phi = 2(\phi' - \phi) = 2\alpha.$$

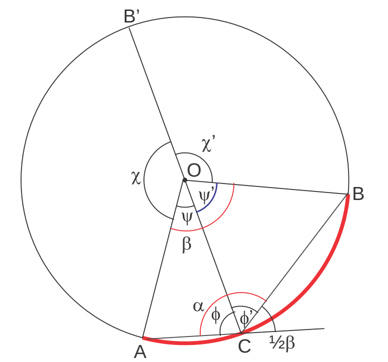


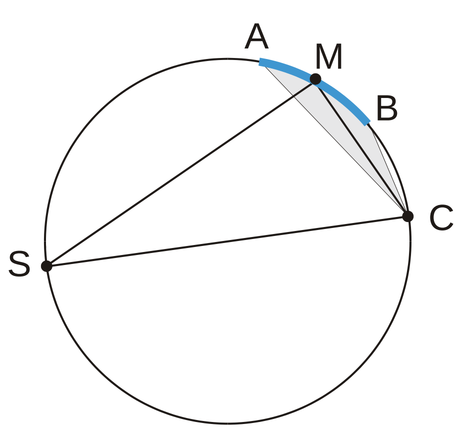
Fig. 4

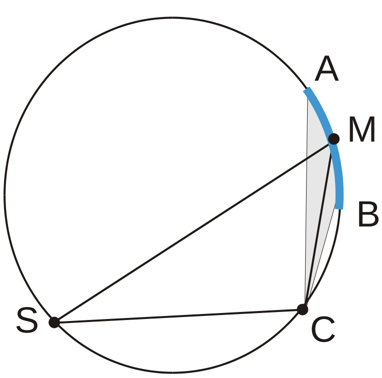
Only the case of obtuse angle is left. In this case the ray CB' passes through the center of the circle and divides angle ∠ACB into two angles ϕ and ϕ' They are not now half of the angles ψ and ψ', but half of their supplement angles χ and χ' therefore,

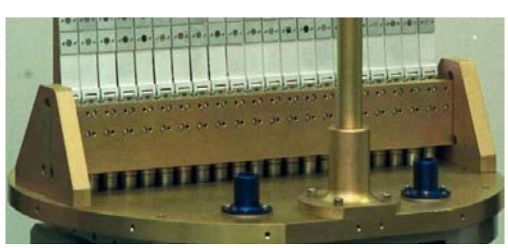
$$\alpha = \frac{1}{2}\chi + \frac{1}{2}\chi' = \frac{1}{2}(\chi + \chi') = \frac{1}{2}(180 - \psi + 180 - \psi') = 180 - \frac{1}{2}(\psi + \psi') = 180 - \frac{1}{2}\beta.$$

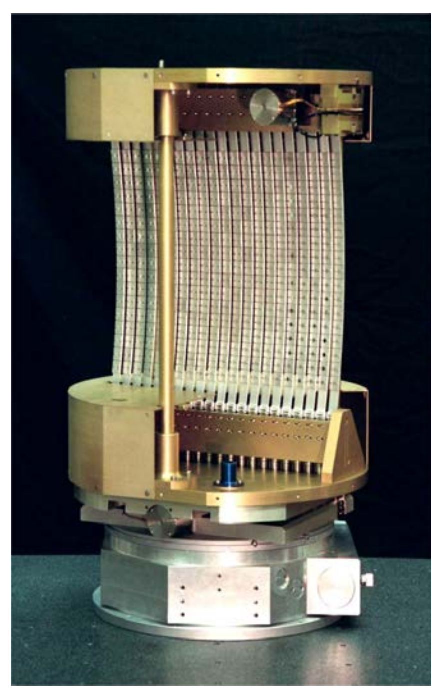
The Rowland circle.

In scientific diffraction instruments it is often desirable to have a diffraction mirror shaped in a way such that the reflection of a beam of light, or particles, emanating from a point source, and focused to a point, corresponds to the same angle between the incident and the reflected (diffracted) beam for any point on the mirror. Such mirror is a segment of the so-called Rowland circle.

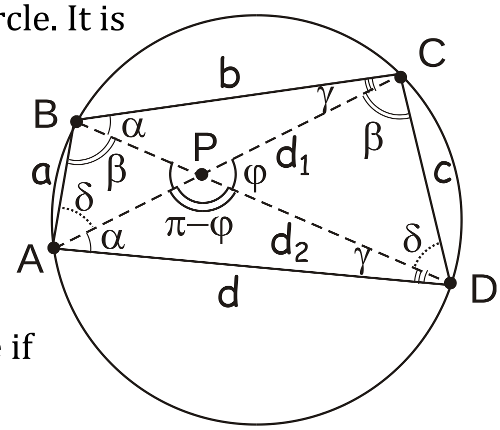








Properties of inscribed quadrilaterals. Ptolemey's theorem.



Consider the quadrilateral $ABCD$ inscribed into a circle. It is clear from the theorem on the inscribed angle that the opposite angles of $ABCD$ are supplementary (i. e. add to 180 degrees),

$$\hat{A} + \hat{C} = \hat{B} + \hat{D} = \pi$$

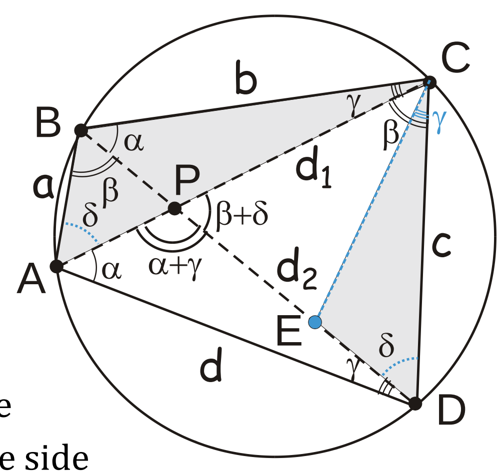
Theorem. A quadrilateral can be inscribed in a circle if and only if its opposite angles are supplementary.

Now consider angles $\alpha, \beta, \gamma, \delta$, between the sides and the diagonals. The angle between the diagonals, $\varphi = \alpha + \gamma = \pi - (\beta + \delta)$.

Theorem (Ptolemey). A quadrilateral can be inscribed in a circle if and only if the product f its diagonals equals the sum of the products of its opposite sides,

$$d_1 d_2 \ = \ ac + bd \qquad (1)$$

<u>Proof of the necessary condition of Ptolemey's theorem</u>, i.e. of Eq. (1) for an inscribed quadrilateral.



Geometrical proof employs an elegant supplementary construct. Inventing such an additional geometrical element is one of the key, most important and powerful methods of geometrical proof.

Draw segment CE, whose endpoint, E, belongs to the diagonal BD, and which is at an angle $\gamma = \widehat{ACB}$ to the side CD. Thus obtained $\Delta DEC \sim \Delta ABC$. Therefore, $\frac{|AC|}{c} = \frac{a}{|ED|}$.

Furthermore, $\widehat{BCE} = \widehat{ACD} = \beta$ and therefore $\Delta BCE \sim \Delta ACD$, so $\frac{|AC|}{d} = \frac{b}{|BE|}$.

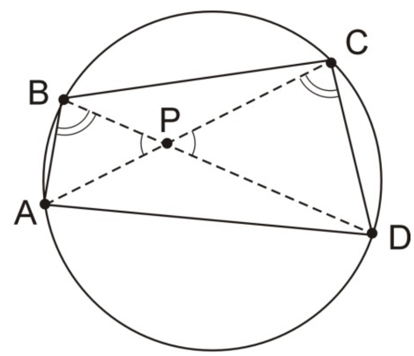
Adding thus obtained equalities, we get

$ac + bd = |AC||ED| + |AC||BE| = d_1 d_2.$

The sufficiency of this condition can be easily proven by contradiction.

Euclids' theorems. Power of a point to a circle.

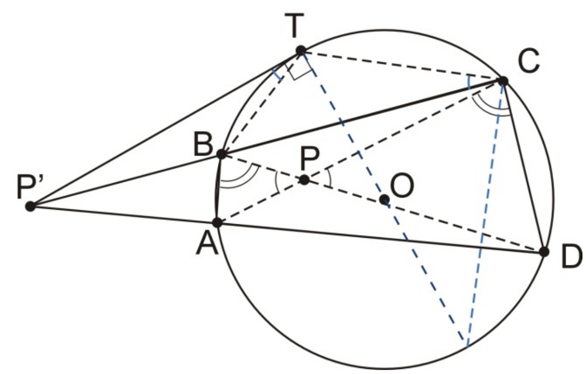
Consider the following figures. Using the theorem on the angle inscribed into a circle and the similarity of the corresponding triangles, it is easy to prove the following Euclid theorems.



i. If two chords AC and BD intersect at a point P inside the circle, then

$$|AP||PC| = |BP||PD| = R^2 - d^2,$$

where R is the radius of the circle and and d is the distance from point P to the center of the circle, $d = |PO|$.



Proof. $\Delta APB \sim \Delta DPC$, so $\frac{|AP|}{|BP|} = \frac{|PD|}{|PC|}$, or, $|AP||PC| = |BP||PD| = R^2 - d^2$.

ii. If two chords AD and BC intersect at a point P' outside the circle, then

$$|P'A||P'D| = |P'B||P'C| = |PT|^2 = d^2 - R^2,$$

where $|PT|$ is a segment tangent to the circle.

Proof. $\Delta P'BD \sim \Delta P'AC$, so $\frac{|P'A|}{|P'B|} = \frac{|P'D|}{|P'C|}$, or, $|P'A||P'D| = |P'B||P'C|$.

For any circle of radius R and any point P distant d from the center of the circle, the quantity $d^2 - R^2$ is called the power of P with respect to the circle.