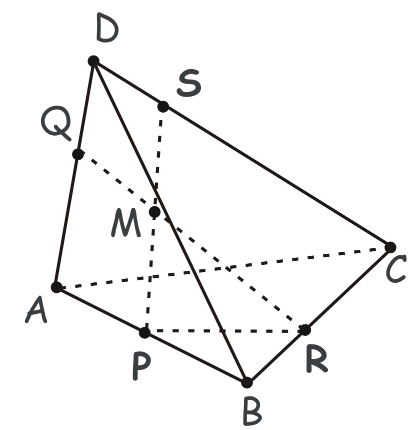
Homework for November 11, 2018.

Geometry.

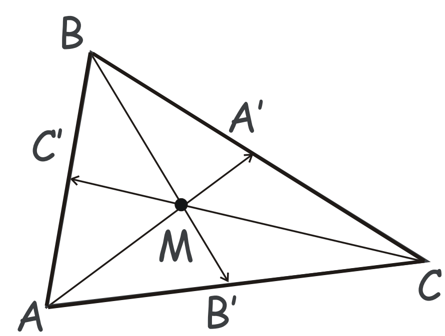
Review the classwork handout. Solve the unsolved problems from previous homework (some are repeated below). Try solving the following problems using the method of point masses and the Law of Lever.

Problems.

1. Each vertex of the tetrahedron $ABCD$ is connected with the centroid of the opposite face (the crossing point of its medians). Prove that all four of these segments, as well as the segments connecting the midpoints of the opposite edges (opposite edges have no common points; there are three pairs of opposite edges in a tetrahedron, and therefore three such segments) – seven segments in total, have common crossing point (are concurrent).



2. In a quadrilateral $ABCD$, E and F are the mid-points of its diagonals, while O is the point where the midlines (segments conneting the midpoints of the opposite sides) cross. Prove that E, F, and O are collinear (belong to the same line).
3. In a triangle ABC, Cevian segments AA', BB' and CC' are concurrent and cross at a point M (point C' is on the side AB, point B' is on the side AC, and point A' is on the side BC). Given the ratios $\frac{AC'}{C'B} = p$ and $\frac{AB'}{B'C} = q$, find the ratio $\frac{AM}{MA'}$ (express it through p and q).



4. What is the ratio of the two segments into which a line passing through the vertex A and the middle of the median BB' of the triangle ABC divides the median CC'?
5. What is the ratio of the two segments into which a line passing through the vertex A and the middle of the median BB' of the triangle ABC divides the median CC'?

6. In a parallelogram $ABCD$, a line passing through vertex D passes through a point E on the side AB, such that $|AE|$ is $1/n$-th of $|AB|$, n is an integer. At what distance from A, relative to the length, $|AC|$, of the diagonal AC it meets this diagonal?

Algebra.

Review the classwork handout and complete the exercises. Solve the remaining problems from the previous homework (you may skip the ones considered in class). Solve the following problems.

1. Using mathematical induction, prove that $\forall n \in \mathbb{N}$,
   a. $\sum_{k=1}^{n}(2k-1)^2 = 1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{4n^3-n}{3}$,
   b. $\sum_{k=1}^{n}(2k)^2 = 2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2n(2n+1)(n+1)}{3}$
   c. $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = (1+2+3+\cdots+n)^2$
   d. $\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} < \frac{1}{2}$
   e. $\sum_{k=1}^{n} \frac{1}{(7k-6)(7k+1)} = \frac{1}{1\cdot 8} + \frac{1}{8\cdot 15} + \frac{1}{15\cdot 22} + \cdots + \frac{1}{(7n-6)(7n+1)} < \frac{1}{7}$
   f. $\sum_{k=n+1}^{3n+1} \frac{1}{k} = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n+1} > 1$

Recap. In order to prove the equality $A(n) = B(n)$, for any n, using the method of mathematical induction you have to

- Prove that $A(1) = B(1)$
- Prove that $A(k+1) - A(k) = B(k+1) - B(k)$ (*)
- Then from assumption $A(k) = B(k)$ and from equality (*) follows $A(k+1) = B(k+1)$

2. Prove by mathematical induction that for any natural number n,
   a. $5^n + 6^n - 1$ is divisible by 10
   b. $9^{n+1} - 8n - 9$ is divisible by 64
3. **Recap.** Binomial coefficients are defined by
   $C_n^k = {}_kC_n = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.
   a. Prove that $C_{n+k}^2 + C_{n+k+1}^2$ is a full square
   b. Find n satisfying the following equation,

   $$C_n^{n-1} + C_n^{n-2} + C_n^{n-3} + \cdots + C_n^{n-10} = 1023$$

   c. Prove that

$$\frac{C_n^1 + 2C_n^2 + 3C_n^3 + \cdots + nC_n^n}{n} = 2^{n-1}$$

4. Find the roots of the equation:

$$\frac{14}{20-6x-2x^2} + \frac{x^2+4x}{x^2+5x} - \frac{x+3}{x-2} + 3 = 0.$$