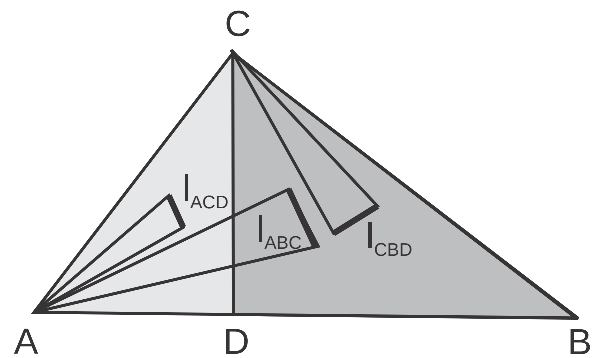
October 14, 2018

Geometry.

<u>Generalized Pythagorean Theorem (continued)</u>.

Theorem 2. For three homologous segments, l_{ABC}, l_{CBD} and l_{ACD} belonging to the similar right triangles ABC, CBD and ACD, where CD is the altitude of the triangle ABC drawn to its hypotenuse AB, the following holds,

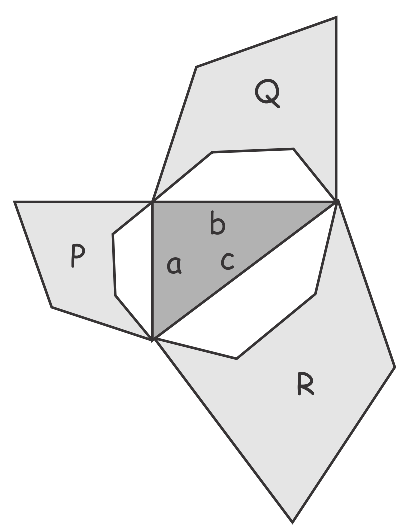


$$l_{ACD}^2 + l_{CBD}^2 = l_{ABC}^2$$

Proof. If we square the similarity relation for the homologous segments, $\frac{l_{CBD}}{a} = \frac{l_{ACD}}{b} = \frac{l_{ABC}}{c}$, where $a = |BC|$, $b = |AC|$ and $c = |AB|$ are the legs and the hypotenuse of the triangle ABC, we obtain, $\frac{l_{CBD}^2}{a^2} = \frac{l_{ACD}^2}{b^2} = \frac{l_{ABC}^2}{c^2}$. Using the property of a proportion, we may then write, $\frac{l_{ACD}^2 + l_{CBD}^2}{a^2+b^2} = \frac{l_{ABC}^2}{c^2}$, wherefrom, by Pythagorean theorem for the right triangle ABC, $a^2 + b^2 = c^2$, we immediately obtain $l_{ACD}^2 + l_{CBD}^2 = l_{ABC}^2$.

Theorem 1. If three similar polygons, P, Q and R with areas S_P, S_Q and S_R are constructed on legs a, b and hypotenuse c, respectively, of a right triangle, then,

$$S_P + S_Q = S_R$$

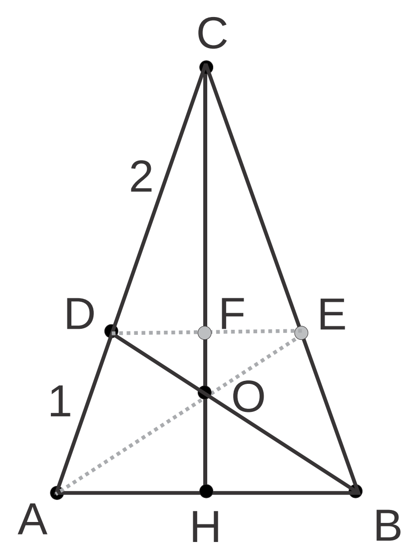


Proof. The areas of similar polygons on the sides of a right triangle satisfy $\frac{S_R}{S_P} = \frac{c^2}{a^2}$ and $\frac{S_R}{S_Q} = \frac{c^2}{b^2}$, or, $\frac{S_P}{a^2} = \frac{S_Q}{b^2} = \frac{S_R}{c^2}$. Using the property of a proportion, we may then write, $\frac{S_P+S_Q}{a^2+b^2} = \frac{S_R}{c^2}$, wherefrom, using the

Pythagorean theorem for the right triangle ABC, $a^2 + b^2 = c^2$, we immediately obtain $S_P + S_Q = S_R$.

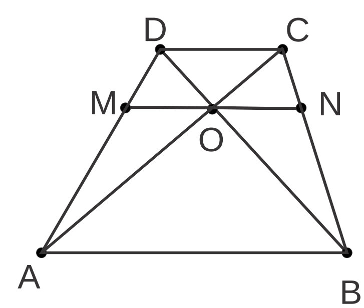
<u>Selected problems on similar triangles</u>.

Problem 1 (homework problem #4). In the isosceles triangle ABC point D divides the side AC into segments such that $|AD|:|CD| = 1:2$. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio $|OH|$ to $|CH|$.



Solution. First, let us perform a supplementary construction by drawing the segment DE parallel to AB, $DE||AB$, where point E belongs to the side CB, and point F to DE and the altitude CH. Notice the similar triangles, $AOH \sim DOF$, which implies, $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|}$. By Thales theorem, $\frac{|AH|}{|DF|} = \frac{|AC|}{|AD|} = 1 + \frac{|CD|}{|AD|} = \frac{3}{2}$, and $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|} = \frac{2}{3}$, so that $\frac{|FH|}{|OH|} = \frac{|FO|+|OH|}{|OH|} = \frac{5}{3}$. $\frac{|CH|}{|OH|} = \frac{|CH|}{|FH|}\frac{|FH|}{|OH|} = 3 \cdot \frac{5}{3} = 5$, because $\frac{|CH|}{|FH|} = 1 + \frac{|CF|}{|FH|} = 1 + \frac{|CD|}{|DA|}$. Therefore, the sought ratio is, $\frac{|OH|}{|CH|} = \frac{1}{5}$.

Problem 2 (homework problem #5). In a trapezoid $ABCD$ with the bases $|AB| = a$ and $|CD| = b$, segment MNparallel to the bases, $MN||AB$, connects the opposing sides, $M \in [AD]$ and $N \in [BC]$. MN also passes through the intersection point O of the diagonals, AC and BD, as shown in the Figure. Prove that $|MN| = \frac{2ab}{a+b}$.



Solution. By Thales theorem applied to vertical angles AOB and DOC and parallel lines AB and CD, $\frac{|AM|}{|MD|} = \frac{|BN|}{|NC|} = \frac{|AB|}{|DC|} = \frac{a}{b}$. Consequently, $\frac{|AD|}{|MD|} = \frac{|AM|+|MD|}{|MD|} = \frac{a}{b} + 1 = \frac{|BN|+|NC|}{|NC|} = \frac{|BC|}{|NC|}$. Now, applying the same Thales theorem to angles ADB and ACB and parallel lines MN and AB, we obtain, $\frac{|MO|}{|AB|} = \frac{|MD|}{|AD|} = \frac{1}{\frac{a}{b}+1}$ and $\frac{|ON|}{|AB|} = \frac{|NC|}{|BC|} = \frac{1}{\frac{a}{b}+1}$. Hence, $\frac{|MO|}{|AB|} + \frac{|ON|}{|AB|} = \frac{|MN|}{|AB|} = \frac{2}{\frac{a}{b}+1}$, and $|MN| = \frac{2ab}{a+b}$.