October 14, 2018

Algebra.

Elements of Mathematical Logic (continued).

Predicate Calculus. Quantifiers.

A **predicate** is a verb phrase that describes a property of objects, or a relationship among objects represented by the variables. We can use notation $P(x, y)$, where P is a predicate name, and x and y denote objects or variables. Informally, a predicate is a statement that may be true or false depending on the values of its variables. Predicate generalizes the concept "proposition".

Predicate Calculus is the branch of formal logic, which deals with representing the logical connections between statements, as well as the statements themselves. The predicate calculus preserves all operations of the propositional calculus, but enlarges it by allowing compound functional and predicate terms and the two quantifiers, ∀ and ∃.

Definition. A **predicate** with a variable is a **proposition**, if either (i), or (ii):

i. a value is assigned to the variable
ii. possible values of the variable are quantified using a **quantifier**

Example. For $x > 1$ to be a proposition, either we must substitute a specific number for x, or change it to something like "There is a number x for which $x > 1$ holds"; equivalently, using a quantifier, $\exists\, x, x > 1$.

Quantifiers.

∃ is called the **existential quantifier**, and reads " … there exists …".

$\exists\, x \in X: \Leftrightarrow$ " … there exists an x in the set X such that …"

For example, "someone lives on a remote island" could be transformed into the propositional form, $\exists\, x: P(x)$, where:

- $P(x)$ is the predicate, stating: **x lives on a remote island**,
- Set of objects of interest X includes (not limited to) all living creatures.

The statement $D(x)$: “equation $x^3 + 3x^2 + 5x + 15 = 0$ has a real solution”, can be written in a predicate form as: $\exists x \in R: x^3 + 3x^2 + 5x + 15 = 0$.

Exercise. Try to construct negation for $P(x)$ and $D(x)$.

$\forall$ is called the **universal quantifier**, and reads “ … for all …”.

$\forall x \in X: \Leftrightarrow$ “ … for all x in the set X …”

Example 1. "All cars have wheels" could be transformed into the propositional form, $\forall\{x, (x\ is\ a\ car)\}: D(x)$, where,

- $D(x)$ is the predicate denoting: **x has wheels**, and
- Set of objects of interest, X, is only populated by cars.

Example 2. $\forall x, x < x^2$. Is this true or false? How we fix it if we should?

These two quantifiers (plus the usual logical operations such as conjunction and disjunction, i.e. AND, OR,...) are sufficient to write all statements in mathematics. This gives rise to a standard mathematical language, which greatly facilitates expressing mathematical reasoning in proofs and problem solving, which we will be using throughout this course.

<u>**Negation with Quantifiers. Predicate Negation Laws.**</u>

Predicate Negation Laws. [Generalized De Morgan]

$\sim(\exists x \in X: p_i) \equiv \forall x \in X: \sim p_i$

$\sim(\forall x \in X: p_i) \equiv \exists x \in X: \sim p_i$

Negation of statements with quantifiers and implications.

1. $(\exists x \in X: P(x)) \equiv$ there exists x in X such that $P(x)$ is satisfied. The negation of it would be,

$\sim(\exists x \in X: P(x)) \equiv$ (It is not the case that there exists x in X such that $P(x)$ is satisfied) $\equiv$ (for any x in X opposite of $P(x)$ is satisfied) $\equiv (\forall x \in X: \sim P(x))$.

2. $(\forall x \in X: P(x)) \equiv$ (for any $x \in X$ $P(x)$ is satisfied). Negation of it would be,

$\sim(\forall x \in X: P(x)) \equiv$ (It is not the case that for any x in X $P(x)$ is satisfied) $\equiv$ (there exists x in X such that $P(x)$ is not satisfied) $\equiv (\exists x \in X: \sim P(x))$.

Example 1. The negation of a proposition (there are positive integers n such that $2^{2^n}+1$ is not a prime) would be a proposition, (for every positive integer n, $2^{2^n}+1$ is a prime),

$\sim(\exists n \in \mathrm{N}:\ 2^{2^n}+1 \text{ is not a prime}) \equiv (\forall n \in N: 2^{2^n}+1 \text{ is a prime})$.

Example 2. The negation of a proposition (Every prime is odd) would be a proposition that not every prime is odd, or, that there exists at least one prime that is even,

$\sim(\forall n, (n \text{ is prime}): (n \text{ is odd})) \equiv (\exists n, (n \text{ is prime}): (n \text{ is even}))$.

In fact, even a stronger proposition holds, $(\exists!\, n, (n \text{ is prime}): (n \text{ is even}))$.

Negation with Multiple Quantifiers.

3. $((\forall x \in X), (\exists y \in Y): P(x,y)) \equiv$ (for all x in X there exists y in Y such that $P(x,y)$ is satisfied). The negation of it would be,

$$\sim\big(((\forall x \in X), (\exists y \in Y): P(x,y)\big) \;\equiv\; \big((\exists x \in X), (\forall y \in Y): \sim P(x,y)\big)$$

Negation of Implications and Equivalencies.

1. $\sim(A \Rightarrow B) \Leftrightarrow \big(A \wedge \sim(B)\big)$

The negation of (A implies B) $\equiv$ (B follows from A) would be a proposition that a conjunction of A and $\sim(B)$ holds, (both A and an opposite of B hold).

2. $\sim(A \Leftrightarrow B) \Leftrightarrow (\sim(A) \Leftrightarrow B)$
3. $\sim(A \Leftrightarrow B) \Leftrightarrow \big(A \Leftrightarrow \sim(B)\big)$

The negation of (A and B are equivalent) would be (A and opposite of B are equivalent), or, (opposite of B and A are equivalent).

Example. The inverse Pythagorean theorem. ($\forall$ triangle ABC with sides a, b, and c, $a^2+b^2=c^2$) $\Rightarrow$(ABC is a right triangle with hypothenuse c and legs a, b).

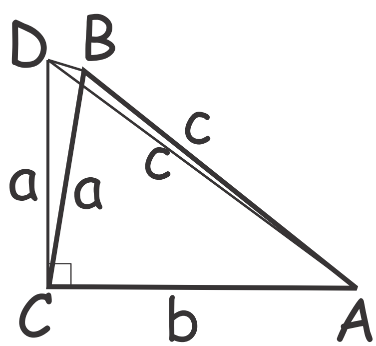
Proof . Proof by contradiction (*reductio ad absurdum*) proceeds by assuming that the opposite to the statement of the theorem is true,

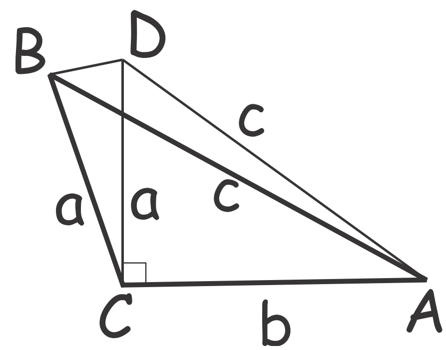
~(($\forall$ triangle ABC with sides a, b, and c, $a^2 + b^2 = c^2$)⇒(ABC is a right triangle with hypothenuse c and legs a, b)), or,

($\exists$ triangle ABC with sides a, b, and c, $a^2 + b^2 = c^2$)∧(ABC is not a right triangle)

One way to obtain the contradiction is illustrated by the auxiliary additional construction shown below (consider the angles $\widehat{CDB} = \widehat{CBD} > \widehat{ADB} = \widehat{ABD}$). There are also other ways.

Exercise. What other proofs can you suggest?





A summary of logical equivalences.

Commutative laws:

1. $(A \wedge B) \Leftrightarrow (B \wedge A)$
2. $(A \vee B) \Leftrightarrow (B \vee A)$
3. $(A \Leftrightarrow B) \Leftrightarrow (B \Leftrightarrow A)$

Associative laws:

1. $\big(A \wedge (B \wedge C)\big) \Leftrightarrow \big((A \wedge B) \wedge C\big)$
2. $\big(A \vee (B \vee C)\big) \Leftrightarrow \big((A \vee B) \vee C\big)$
3. $\big(A \Leftrightarrow (B \Leftrightarrow C)\big) \Leftrightarrow \big((A \Leftrightarrow B) \Leftrightarrow C\big)$

Distributive laws:

4. $\big(A \wedge (B \vee C)\big) \Leftrightarrow \big((A \wedge B) \vee (A \wedge C)\big)$
5. $\big(A \vee (B \wedge C)\big) \Leftrightarrow \big((A \vee B) \wedge (A \vee C)\big)$
6. $\big(A \Rightarrow (B \wedge C)\big) \Leftrightarrow \big((A \Rightarrow B) \wedge (A \Rightarrow C)\big)$
7. $\big(A \Rightarrow (B \vee C)\big) \Leftrightarrow \big((A \Rightarrow B) \vee (A \Rightarrow C)\big)$
8. $\big((A \wedge B) \Rightarrow C\big) \Leftrightarrow \big((A \Rightarrow C) \vee (B \Rightarrow C)\big)$
9. $\big((A \vee B) \Rightarrow C\big) \Leftrightarrow \big((A \Rightarrow C) \wedge (B \Rightarrow C)\big)$

Negation laws:

4. $\sim(A \wedge B) \Leftrightarrow \big(\sim(A) \vee \sim(B)\big)$
5. $\sim(A \vee B) \Leftrightarrow \big(\sim(A) \wedge \sim(B)\big)$
6. $\sim(\sim A) \Leftrightarrow A$
7. $\sim(A \Rightarrow B) \Leftrightarrow \big(A \wedge \sim(B)\big)$
8. $\sim(A \Leftrightarrow B) \Leftrightarrow (\sim(A) \Leftrightarrow B)$
9. $\sim(A \Leftrightarrow B) \Leftrightarrow \big(A \Leftrightarrow \sim(B)\big)$

Implication laws:

1. $(A \Rightarrow B) \Leftrightarrow \Big(\sim\big(A \wedge \sim(B)\big)\Big)$
2. $(A \Rightarrow B) \Leftrightarrow (\sim(A) \vee B)$
3. $(A \Rightarrow B) \Leftrightarrow \big(\sim(B) \Rightarrow \sim(A)\big)$
4. $(A \Leftrightarrow B) \Leftrightarrow \big((A \Rightarrow B) \wedge (B \Rightarrow A)\big)$
5. $(A \Leftrightarrow B) \Leftrightarrow \big(\sim(A) \Leftrightarrow \sim(B)\big)$

Recap. Properties of rational numbers ($\mathbb{Q}$) and algebraic operations.

Ordering and comparison.

1. $\forall\, a, b \in \mathbb{Q}$, one and only one of the following relations holds
   - $a = b$
   - $a < b$
   - $a > b$
2. $\forall\, a, b \in \mathbb{Q}, \exists c \in \mathbb{Q}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity. $\forall\, a, b, c \in \mathbb{Q}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property. $\forall\, a, b \in \mathbb{Q}, a > b > 0, \exists n \in \mathbb{N}$, such that $a < nb$

Addition and subtraction.

- $\forall\, a, b \in \mathbb{Q}, a + b = b + a$
- $\forall\, a, b, c \in \mathbb{Q}, (a + b) + c = a + (b + c)$
- $\forall\, a \in \mathbb{Q}, \exists 0 \in \mathbb{Q}, a + 0 = a$
- $\forall\, a \in \mathbb{Q}, \exists - a \in \mathbb{Q}, a + (-a) = 0$
- $\forall\, a, b \in \mathbb{Q}, a - b = a + (-b)$
- $\forall\, a, b, c \in \mathbb{Q}, (a < b) \Rightarrow (a + c < b + c)$

Multiplication and division.

- $\forall\, a, b \in \mathbb{Q}, a \cdot b = b \cdot a$
- $\forall\, a, b, c \in \mathbb{Q}, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- $\forall\, a, b, c \in \mathbb{Q}, (a + b) \cdot c = a \cdot c + b \cdot c$
- $\forall\, a \in \mathbb{Q}, \exists 1 \in \mathbb{Q}\,, a \cdot 1 = a$
- $\forall\, a \in \mathbb{Q}, a \neq 0, \exists \frac{1}{a} \in \mathbb{Q}, a \cdot \frac{1}{a} = 1$
- $\forall\, a, b \in \mathbb{Q}, b \neq 0, \frac{a}{b} = a \cdot \frac{1}{b}$
- $\forall\, a, b, c \in \mathbb{Q}$, c>0, $(a < b) \Rightarrow (a \cdot c < b \cdot c)$
- $\forall\, a \in \mathbb{Q}, a \cdot 0 = 0, a \cdot (-1) = -a$

<u>Logical fallacies</u>

A fallacy is reasoning that is evaluated as logically incorrect. Fallacy vitiates the logical validity of the argument and warrants its recognition as unsound.

<u>Formal fallacies</u>

A formal fallacy is an error in logic that can be seen in the argument's form. All formal fallacies are specific types of *non sequiturs* (does not follow).

- Appeal to probability – is a statement that takes something for granted because it would probably be the case (or might be the case).
- Argument from fallacy – also known as fallacy fallacy, assumes that if an argument for some conclusion is fallacious, then the conclusion is false.

If you are paranoid about being stalked does not mean you are not stalked.

- Base rate fallacy – making a probability judgment based on conditional probabilities, without accounting for the effect of prior probabilities.
- Conjunction fallacy – assumption that an outcome simultaneously satisfying multiple conditions is more probable than an outcome satisfying a single one of them.
- Masked-man fallacy (illicit substitution of identicals) – the substitution of identical designators in a true statement can lead to a false one. *I know how to solve math problems; I don't know whether this is a math problem => I don't know how to solve this problem.*
- Jumping to conclusions – the act of taking decisions without having enough information to be sure they are right.

Propositional fallacies

A propositional fallacy is an error in logic that concerns compound propositions. For a compound proposition to be true, the truth values of its constituent parts must satisfy the relevant logical connectives that occur in it (most commonly: <and>, <or>, <not>, <only if>, <if and only if>). The following fallacies involve inferences whose correctness does not follow from the properties of those logical connectives, and hence, which are not guaranteed to yield logically true conclusions.

- Affirming a disjunct – A or B; A, therefore not B.
- Affirming the consequent –if A, then B; B, therefore A.
- Denying the antecedent – if A, then B; not A, therefore not B.

Quantification fallacies

A quantification fallacy is an error in logic where the quantifiers of the premises are in contradiction to the quantifier of the conclusion.

- Existential fallacy – an argument that has a universal premise and a particular conclusion. "In a communist society everyone has everything (s)he needs”, or, “In a communist society everyone suffers from oppression”, or, “Every Unicorn has one horn on its forehead”.

- A vacuous truth is a conditional statement with a false antecedent. A statement that asserts that all members of the empty set have a certain property. For example, the statement "all students in the room are in math 9 class" will be true whenever there are no students in the room. In this case, the statement "all students in the room are not in math 9 class" would also be vacuously true, as would the conjunction of the two: "all students in the room are in Math 9 and are not in Math 9”.

Syllogistic fallacies – logical fallacies that occur in syllogisms.

- Affirmative conclusion from a negative premise (illicit negative) – when a categorical syllogism has a positive conclusion, but at least one negative premise. “Smart people don’t eat junk food. I do not eat junk food. Therefore, I a smart”.
- Fallacy of exclusive premises – a categorical syllogism that is invalid because both of its premises are negative.
- Fallacy of four terms (quaternio terminorum) – a categorical syllogism that has four terms. *Nothing is better than eternal happiness; ham sandwich is better than nothing => ham sandwich is better than eternal happiness*.
- Illicit major – a categorical syllogism that is invalid because its major term is not distributed in the major premise but distributed in the conclusion. All A are B; No C are A. Therefore, no C are B.
- Illicit minor – a categorical syllogism that is invalid because its minor term is not distributed in the minor premise but distributed in the conclusion. Pie is good. Pie is unhealthy. Thus, all good things are unhealthy.
- Negative conclusion from affirmative premises (illicit affirmative) – when a categorical syllogism has a negative conclusion but affirmative premises. All A is B. All B is C. Hence, some C is not A.
- Fallacy of the undistributed middle – the middle term in a categorical syllogism is not distributed. All Z is B; All Y is B. Therefore, all Y is Z.

- Modal fallacy – confusing possibility with necessity.

Informal fallacies

Informal fallacies – arguments that are fallacious for reasons other than structural (formal) flaws and usually require examination of the argument's content.

- Appeal to the stone (argumentum ad lapidem) – dismissing a claim as absurd without demonstrating proof for its absurdity.
- …
- Correlation proves causation (post hoc ergo propter hoc)
- Divine fallacy (argument from incredulity) – arguing that, because something is so incredible/amazing/ununderstandable, it must be the result of superior, divine, alien or paranormal agency.
- Double counting – counting events or occurrences more than once in probabilistic reasoning, which leads to the sum of the probabilities of all cases exceeding unity.
- Equivocation – the misleading use of a term with more than one meaning (by glossing over which meaning is intended at a particular time).
- …
- Psychologist's fallacy – an observer presupposes the objectivity of his own perspective when analyzing a behavioral event.
- Red herring – a speaker attempts to distract an audience by deviating from the topic at hand by introducing a separate argument the speaker believes is easier to speak to.
- Referential fallacy – assuming all words refer to existing things and that the meaning of words reside within the things they refer to, as opposed to words possibly referring to no real object or that the meaning of words often comes from how we use them.
- …

<u>Principle of Mathematical induction</u>.

Let $\{P(n)\} = P(1), P(2), P(3), \ldots$ be a sequence of propositions numbered by positive integers, which together constitute the general theorem P. In particular, $P(n)$ can be some formula, or other property of positive integers. Suppose that by some mathematical argument it can be shown that,

(1) <u>Base Case</u>: $P(1)$ is true, and

(2) <u>Inductive Step</u>: **if** $P(n)$ is true, **then** $P(n+1)$ is true: $P(n) \Rightarrow P(n+1)$.

Then, $P(n)$ is true for all positive integers: $\forall n, P(n)$, so all the propositions of the sequence are true and the theorem P is proved.

The principle of mathematical induction rests on the fact that after any integer, n, there is a next one, n+1, and that any integer can be achieved by a finite number of steps incrementing the previous integer by 1, starting from 1.

Although logically obvious, the principle of mathematical induction can be proven as a mathematical theorem using the "principle of smallest integer", which states: **every non-empty set *S* of positive integers has a smallest number**. Indeed, S must contain at least one integer, say n, and the smallest of integers $1,2, \ldots, n$ belonging to S will be the smallest integer in it.

Consider a sequence of statements $\{P(n)\} = P(1), P(2), P(3)$, …, such that,

- $P(1)$ is true, and
- For any positive integer if $P(n)$ is true, then $P(n+1)$ is true: $\forall n \in \mathbb{N}, P(n) \Rightarrow P(n+1)$.

Let us assume that one of the statements in $P = \{P(n)\}$ is false: $\exists m \in \mathbb{N}, \sim\big(P(m)\big)$ and show that such hypothesis is untenable. Indeed, in such case the set of all positive integers for which $P(n)$ is false is non-empty, and therefore has the smallest number, r. Then, $P(r)$ is false while $P(r-1)$ is true, $\Big(\exists r \in \mathbb{N}, \Big(P(r-1) \wedge \sim\big(P(r)\big)\Big)\Big) \Leftrightarrow \sim(\forall r \in \mathbb{N}, P(r) \Rightarrow P(r+1))$. This contradicts our assumption, which completes the proof. ¤