

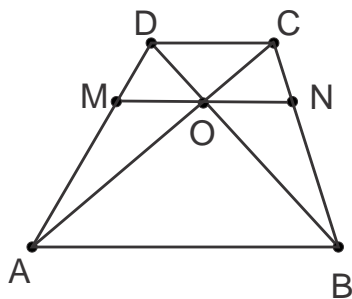
Homework for October 14, 2018.

### Geometry.

Review the classwork handout. Please pay attention to the proof of “inverse Thales theorem”. Solve the remaining problems from the previous homework. In particular, review constructing the segments equal to the ratio and the product of two given segments. Try solving the following additional problems. In all the problems, you are only allowed to use theorems we had proven before!

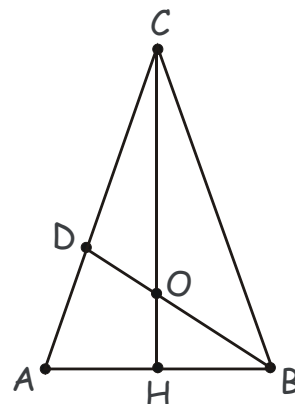
### Problems.

1. Midsegment of a triangle is a line segment joining the midpoints of two sides. Prove that midsegment of a triangle is parallel to the third side and its length is  $1/2$  of that side's length.
2. Using a ruler and a compass, construct a triangle given the midpoints of its three sides.
3. Prove that medians of a triangle divide one another in the ratio 2:1, in other words, the medians of a triangle “trisection” one another (Coxeter, Gretzer, p.8).
4. In isosceles triangle  $ABC$  point  $D$  divides the side  $AC$  into segments such that  $|AD|:|CD|=1:2$ . If  $CH$  is the altitude of the triangle and point  $O$  is the intersection of  $CH$  and  $BD$ , find the ratio  $|OH|$  to  $|CH|$ .
5. In a trapezoid  $ABCD$  with the bases  $|AB| = a$  and  $|CD| = b$ , segment  $MN$  parallel to the bases,  $MN \parallel AB$ ,



connects the opposing sides,  $M \in [AD]$  and  $N \in [BC]$ .  $MN$  also passes through the intersection point  $O$  of the diagonals,  $AC$  and  $BD$ , as shown in the Figure.

Prove that  $|MN| = \frac{2ab}{a+b}$ .



# Algebra.

Review the classwork handout and complete the exercises. Solve the remaining problems from the previous homework. Solve the following problems (you may skip the ones considered in class).

1. Let  $P$ ,  $Q$  and  $R$  be some logical propositions. Write a negation of and simplify the logical expression
  - a.  $((P \wedge Q) \vee (R \wedge Q))$
  - b.  $((P \vee R) \wedge (Q \vee R))$
  - c.  $(P \Leftrightarrow (Q \wedge R))$
  - d.  $((P \wedge Q) \Rightarrow (Q \vee R))$
  - e.  $((P \vee Q) \Leftrightarrow (Q \vee R))$
2. Solve the following equations:
  - a.  $\frac{x-a}{x-b} + \frac{x-b}{x-a} = 2.5$
  - b.  $\sqrt{3x+4} + \sqrt{x-2} = 2\sqrt{x}$
  - c.  $\frac{1}{x^3+2} - \frac{1}{x^3+3} = \frac{1}{12}$
  - d.  $\frac{1}{x^2} + \frac{1}{(x+2)^2} = \frac{10}{9}$
  - e.  $\sqrt{x-2} = x-4$
  - f.  $1 + \sqrt{1 + x\sqrt{x^2 - 24}} = x$
  - g.  $\sqrt{x} + \frac{2x+1}{x+2} = 2$
3. Simplify expressions:
  - a.  $\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}$
  - b.  $\frac{\sqrt{\sqrt{\frac{x-1}{x+1}} + \sqrt{\frac{x+1}{x-1}} - 2}}{\sqrt{(x+1)^3} - \sqrt{(x-1)^3}} (2x + \sqrt{x^2 - 1})$
  - c.  $\frac{\sqrt{x-4\sqrt{x-4}+2}}{\sqrt{x+4\sqrt{x-4}-2}}$
4. Prove that:
  - a.  $\frac{\sqrt{7+4\sqrt{3}} \cdot \sqrt{19-8\sqrt{3}}}{4-\sqrt{3}} - \sqrt{3} = 2$
  - b.  $\sqrt{6a + 2\sqrt{9a^2 - b^2}} - \sqrt{6a - 2\sqrt{9a^2 - b^2}} = 2\sqrt{3a - b}$
5. **Recap.** There are 15 students in a class.

- a. Each student needs to make a presentation of a problem. How many ways are there to arrange the order in which they do presentations – i. e. decide who speaks first, who speaks second, third, ..., fifteenth?
  - b. How many ways is there to select a pair of students of whom one will present solution of an algebra problem, and the other of a problem in geometry?
  - c. How many ways is there to select a team of two students who will represent the class at a math competition?
6. **Recap.** Binomial coefficients are defined by

$$C_n^k = {}_k C_n = \binom{n}{k} = \frac{n!}{k! (n-k)!}$$

Prove that binomial coefficients satisfy the following identities,

$$C_n^0 = C_n^n \Leftrightarrow \binom{n}{0} = \binom{n}{n} = 1$$

$$C_n^k = C_n^{n-k} \Leftrightarrow \binom{n}{k} = \binom{n}{n-k}$$

$$C_{n+1}^{k+1} = C_n^k + C_n^{k+1} \Leftrightarrow \binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

$$C_n^k = C_{n-1}^{k-1} + C_{n-1}^k \Leftrightarrow \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$C_{n+1}^k = C_n^k + C_n^{k-1} \Leftrightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$C_n^{k+1} = \binom{n}{k+1} = \binom{n}{k} \frac{n-k}{k+1}$$

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k} + \cdots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$