MATH 7. HANDOUT 26: INTRODUCTION TO CALCULUS

In this handout we will introduce three major concepts of calculus we talked about in class:

1. Limits
2. Derivatives
3. Integrals

1. Limits

Consider the sequence of numbers

$$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{4}, \quad a_5 = \frac{1}{5}, \ldots.$$

We can write this sequence in a general form as

$$a_n = \frac{1}{n}$$

What happens to these numbers as n becomes larger and larger? It is easy to see that these numbers become closer and closer to 0, but never reach 0. In fact, we can get as close to 0 as we want by choosing a large enough n. For example, if we want to be 0.001 away from 0, we can choose $n = 1000$:

$$a_{1000} = \frac{1}{1000} = 0.001$$

All further members of the sequence will only get closer to 0.

In this case we say that the **limit** of the sequence a_n is 0 as n goes to ∞ (infinity). The way to write it is:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

Now let us consider another sequence:

$$b_n = \frac{2n+1}{n+2}$$

What would be the limit of this sequence as n gets larger and larger ("approaches ∞")? We cannot answer this question right away yet: The numerator of b_n is becoming larger, but so does the denominator.

Let us use the following trick: divide both numerator and denominator of the expression for b_n by n (or, which is the same, multiply them by $\frac{1}{n}$):

$$\begin{aligned} b_n &= \frac{2n+1}{n+2} \\ &= \frac{\frac{1}{n}\cdot(2n+1)}{\frac{1}{n}\cdot(n+2)} \\ &= \frac{\frac{2n+1}{n}}{\frac{n+2}{n}} \\ &= \frac{2+\frac{1}{n}}{1+\frac{2}{n}} \end{aligned}$$

Now we can notice, that fractions in the numerator and denominator ($\frac{1}{n}$ and $\frac{2}{n}$) are becoming smaller and smaller as n increases, and they influence the value of b_n less and less. Choosing large values of n, the numerator becomes very close to 2, and the denominator becomes very close to 1.

As a result, the value of b_n gets closer and closer to $2/1 = 2$. Mathematically speaking, it means that the limit of b_n is equal to 2:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2n+1}{n+2} = 2$$

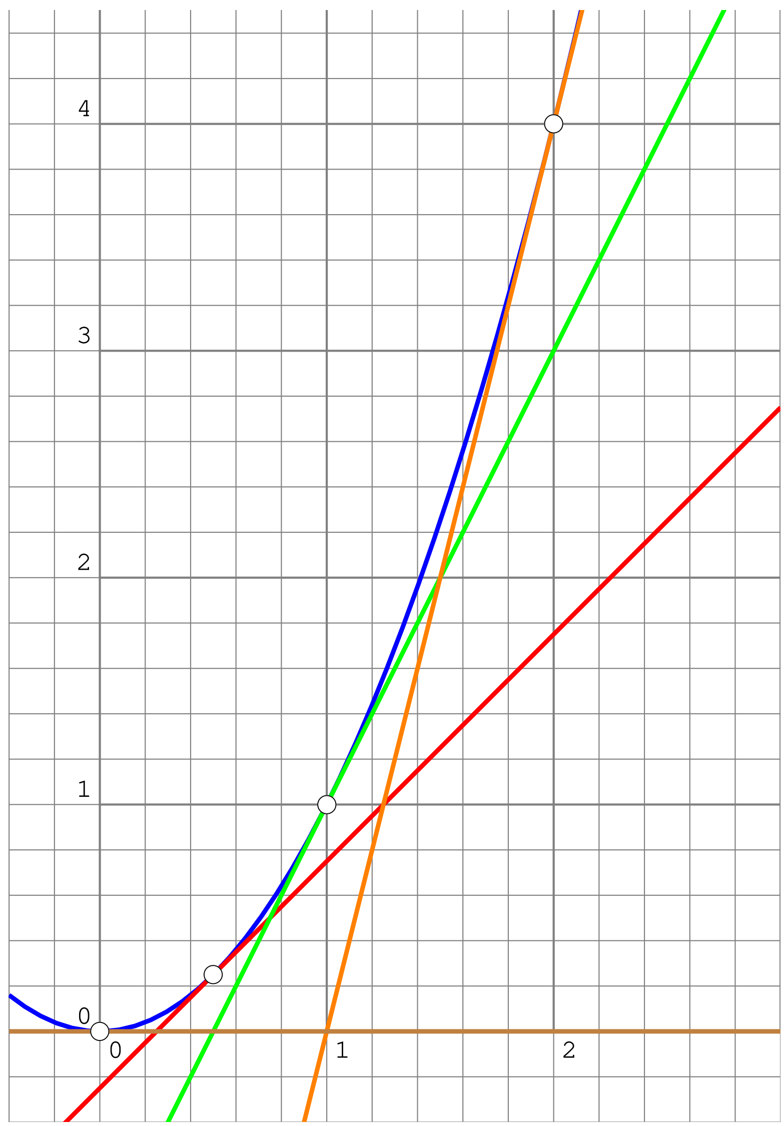
We can see it by computing some members of the sequence:

$$b_1 = \frac{3}{3}, \quad b_2 = \frac{5}{4}, \quad b_3 = \frac{7}{5}, \quad b_4 = \frac{9}{6}, \quad b_5 = \frac{11}{7}, \quad \ldots, \quad b_{20} = \frac{41}{22}, \quad \ldots.$$

2. Derivatives

The next question calculus is interested in is finding slopes of tangent lines.

Tangent line to a graph at a certain point is a line which only "touches" the graph, but does not intersect it (near the given point). Some tangent lines to a graph of $y = x^2$ are given in the following figure (at points $x = 0, x = 0.5, x = 1, x = 2$).



How can we find slopes of these tangent lines? In what follows, I demonstrate how to find a slope of the tangent line to $y = x^2$ at point $x = 1, y = 1$ — i.e. the slope of the green line on the figure above.

Remember how slope is defined? You take the change in y and divide it by the change in x:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Let us do the similar computations here, noticing that $y = x^2$ (the graph we are exploring). Also notice that $x_1 = 1$ and $y_1 = 1$ — this is the point where we want to determine the slope of the tangent line. Now let us try changing x a little, and observe what happens to y as we change x.

Let us start with changing x_1 by 1: $x_2 = x_1 + 1 = 1 + 1 = 2$. In this case, $y_2 = x_2^2 = 2^2 = 4$, and the y-change is $y_2 - y_1 = 2$. Similarly, if we change x_1 by 1/2, we get the following:

$$\begin{aligned} x_2 &= x_1 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}, \\ y_2 &= x_2^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}, \\ y_2 - y_1 &= \frac{9}{4} - 1 = \frac{3}{4}. \end{aligned}$$

Now let's change x_1 by 1/3; we get the following:

$$\begin{aligned} x_2 &= x_1 + \frac{1}{3} = 1 + \frac{1}{3} = \frac{4}{3}, \\ y_2 &= x_2^2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9}, \\ y_2 - y_1 &= \frac{16}{9} - 1 = \frac{7}{9}. \end{aligned}$$

Table below also shows similar computations as we change x by 1/4 and 1/5. In addition, the last column shows the ratio of change in y to change in x:

change in x	x_2	y_2	change in y	$\frac{\text{change in } y}{\text{change in } x}$
1	2	$2^2 = 4$	$4 - 1 = 3$	$3 \div 1 = 3$
$\frac{1}{2}$	$\frac{3}{2}$	$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$	$\frac{9}{4} - 1 = \frac{5}{4}$	$\frac{5}{4} \div \frac{1}{2} = \frac{5}{2}$
$\frac{1}{3}$	$\frac{4}{3}$	$\left(\frac{4}{3}\right)^2 = \frac{16}{9}$	$\frac{16}{9} - 1 = \frac{7}{9}$	$\frac{7}{9} \div \frac{1}{3} = \frac{7}{3}$
$\frac{1}{4}$	$\frac{5}{4}$	$\left(\frac{5}{4}\right)^2 = \frac{25}{16}$	$\frac{25}{16} - 1 = \frac{9}{16}$	$\frac{9}{16} \div \frac{1}{4} = \frac{9}{4}$
$\frac{1}{5}$	$\frac{6}{5}$	$\left(\frac{6}{5}\right)^2 = \frac{36}{25}$	$\frac{36}{25} - 1 = \frac{11}{25}$	$\frac{11}{25} \div \frac{1}{5} = \frac{11}{5}$

Now let us explore what the sequence of slopes goes to as we change x by $\frac{1}{n}$.

$$\begin{aligned}
&x_1 = 1, \qquad y_1 = 1,\\
&x_2 = x_1 + \frac{1}{n} = 1 + \frac{1}{n} = \frac{n+1}{n},\\
&y_2 = x_2^2 = \left(\frac{n+1}{n}\right)^2 = \frac{n^2+2n+1}{n^2},\\
&y_2 - y_1 = \frac{n^2+2n+1}{n^2} - 1 = \frac{n^2+2n+1}{n^2} - \frac{n^2}{n^2} = \frac{2n+1}{n^2}\\
&\frac{y_2-y_1}{x_2-x_1} = \frac{2n+1}{n^2} \div \frac{1}{n} = \frac{2n+1}{n^2} \times n = \frac{2n+1}{n}
\end{aligned}$$

What happens with the last expression as n becomes larger and larger (i.e. $n \to \infty$), and the change in x (which is equal to $1/n$) becomes smaller and smaller? Using the trick from the previous section, we can divide both numerator and denominator by n. We will get:

$$\frac{y_2-y_1}{x_2-x_1} = \frac{2n+1}{n} = \frac{\frac{1}{n}\cdot(2n+1)}{\frac{1}{n}\cdot n} = \frac{2+\frac{1}{n}}{1} = 2 + \frac{1}{n}.$$

Now, as $n \to \infty$, the fraction $\dfrac{1}{n} \to 0$, and the expression above becomes closer and closer to 2:

$$\lim_{n\to\infty} \frac{2n+1}{n} = 2.$$

This is the value of the slope of the tangent line to the graph of $y = x^2$ at point $x = 1, y = 1$ (the green line on the graph above).

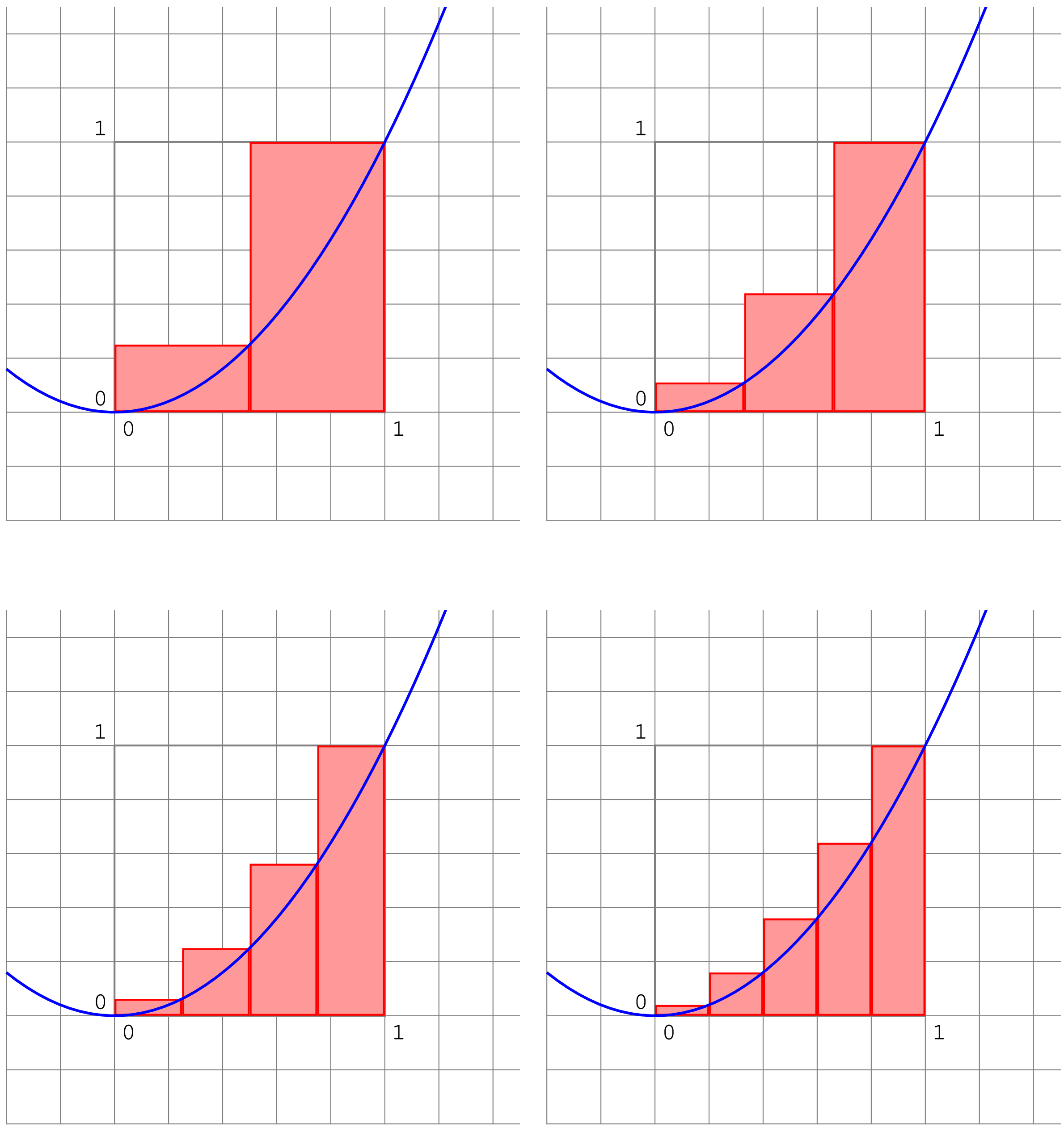
The word **derivative** refers to the slope of the tangent line to the graph, and is denoted by an apostrophe (number 1 in $y'(1)$ refers to the value of x where the tangent line touches the graph):

$$y = x^2; \quad y'(1) = 2$$

3. Integral

Now let us consider the question of computing areas under graphs. For example, consider a parabola, and let us try to figure out what is the area delimited by the parabola $y = x^2$ on top, x-axis on the bottom, and between $x = 0$ and $x = 1$.

To do that we will approximate the area by vertical bars such as the ones shown on the graphs below (from 2 bars to 5 bars). We can break the interval $[0, 1]$ into as many bars as we want, and the more bars we have, the closer we will be getting to the area we are looking for.



Let us compute the shaded bar areas in each of these cases. Notice that for each of these cases the width of each bar is $1/n$, where n is the number of bars.

In case of 2 bars, the area is equal to:

$$\begin{aligned} A_2 &= \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{2} \cdot \left(\frac{2}{2}\right)^2 \\ &= \frac{1^2}{2^3} + \frac{2^2}{2^3} \\ &= \frac{1^2 + 2^2}{2^3} \end{aligned}$$

In case of 3 bars, the area is equal to:

$$\begin{aligned}A_3 &= \frac{1}{3}\cdot\left(\frac{1}{3}\right)^2 + \frac{1}{3}\cdot\left(\frac{2}{3}\right)^2 + \frac{1}{3}\cdot\left(\frac{3}{3}\right)^2 \\ &= \frac{1^2}{3^3} + \frac{2^2}{3^3} + \frac{3^2}{3^3} \\ &= \frac{1^2+2^2+3^2}{3^3}\end{aligned}$$

Similarly, for 4 and 5 bars, we get:

$$\begin{aligned}A_4 &= \frac{1^2+2^2+3^2+4^2}{4^3} \\ A_5 &= \frac{1^2+2^2+3^2+4^2+5^2}{5^3}\end{aligned}$$

Generalizing it to the case of n bars, we arrive at the following formula:

$$A_n = \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$$

To get a better idea of what A_n is, we will use the following formula (which we are not going to prove at the moment):

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Substituting the expression for the sum of squares into the expression for A_n, we get:

$$\begin{aligned}A_n &= \frac{\frac{n(n+1)(2n+1)}{6}}{n^3} \\ &= \frac{n(n+1)(2n+1)}{6n^3} \\ &= \frac{(n+1)(2n+1)}{6n^2} \\ &= \frac{2n^2+n+2n+1}{6n^2} \\ &= \frac{2n^2+3n+1}{6n^2}\end{aligned}$$

What happens to this expression as n becomes larger and larger (i.e. $n \to \infty$)? Let's multiply both numerator and denominator by $\dfrac{1}{n^2}$ (similar to the trick we did before).

$$\begin{aligned}A_n &= \frac{2n^2+3n+1}{6n^2} \\ &= \frac{\frac{1}{n^2}(2n^2+3n+1)}{\frac{1}{n^2}\cdot 6n^2} \\ &= \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}\end{aligned}$$

Now, as $n \to \infty$, fractions $3/n$ and $1/n^2$ in the numerator are becoming smaller and smaller, and influence the answer less and less. As a result, as $n \to \infty$, the value of A_n approaches $2/6 = 1/3$.

$$\lim_{n\to\infty} A_n = \frac{1}{3}.$$

That is exactly the value of the area underneath the parabola we were interested in. This area is often called an **integral**, and this particular area is written mathematically as

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

Homework

1. Find the following limits:

(a) $\lim_{n\to\infty} \frac{n+5}{3n+10}$

(b) $\lim_{n\to\infty} \frac{2n^2+3}{n^2+1}$

(c) $\lim_{n\to\infty} \frac{100n+1}{2n^2+10}$

(d) $\lim_{n\to\infty} \frac{5n^3+2n^2+4n+1}{3n^3+4n^2+7}$

2. (a) Find the slope of the tangent line to the graph of $y = x^2$ at the point $x = 2, y = 4$.

(b) Do the same at the point $x = 3, y = 9$.

*(c) Do you see any pattern here? Can you do a similar computation for a point $x = a, y = a^2$ in general?

3. Find the area underneath the graph of $y = x^3$ between $x = 0$ and $x = 1$. You will need to use the following formula:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$