# Math 7: Handout 25
# Euclidean Geometry – 4.

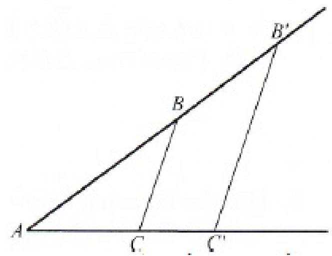
## 1 *Similar triangles*

We say that triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar with coefficient $k$ if $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, $\angle C = \angle C'$ and

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC} = k.$$

We will use notation $\triangle ABC \sim \triangle A'B'C'$.

**Theorem** Consider a triangle $\triangle ABC$ and let $B' \in \overrightarrow{AB}$, $C' \in \overrightarrow{AC}$ be such that lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel. Then $\triangle ABC \sim \triangle A'B'C'$.




**Theorem** For any triangle $\triangle ABC$ and a real number $k > 0$, there exists a triangle $\triangle A'B'C'$ similar to $\triangle ABC$ with coefficient $k$.

**Theorem** [Similarity via AA] Let $\triangle ABC$, $\triangle A'B'C'$ be such that $m\angle A = m\angle A'$, $m\angle B = m\angle B'$. Then these triangles are similar.

**Proof** Let $k = \frac{A'B'}{AB}$. Construct a triangle $\triangle A''B''C''$ which is similar to $\triangle ABC$ with coefficient $k$. Then $A'B' = A''B''$, and $m\angle A = m\angle A' = m\angle A''$, $m\angle B = m\angle B' = m\angle B''$. Thus, by ASA, $\triangle A'B'C' \cong \triangle A''B''C''$.

**Theorem** [Similarity via SAS] Let $\triangle ABC$, $\triangle A'B'C'$ be such that $\angle A = \angle A'$, $\frac{A'B'}{AB} = \frac{A'C'}{AC}$. Then these triangles are similar.

**Theorem** [Similarity via SSS] Let $\triangle ABC$, $\triangle A'B'C'$ be such that

$$\frac{A'B'}{AB} = \frac{A'C'}{AC} = \frac{B'C'}{BC}$$

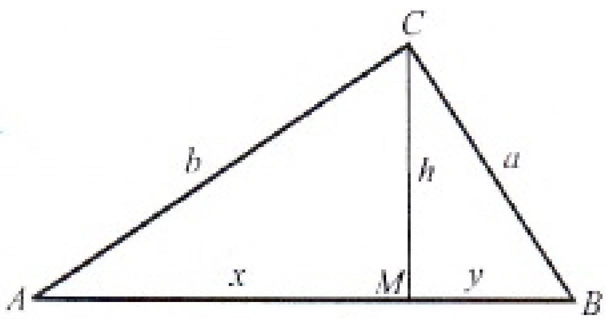
Then these triangles are similar.

One of the most important applications of the theory of similar triangles is to the study of right triangles and Pythagorean theorem.

A right triangle is a triangle in which one of the angles is a right angle. A hypotenuse is the side opposing the right angle; two other sides are called legs.

**Theorem** Let $\triangle ABC$ be a right triangle, with $\angle C$ being the right angle. Let $CM$ be the altitude of angle $C$. Then triangles $\triangle ABC$, $\triangle ACM$, $\triangle CBM$ are all similar.

**Proof** It immediately follows from AA similarity rule.




This theorem immediately implies a number of important relations between various lengths in these triangles. We will give one of them. Denote for brevity $a = BC$, $b = AC$, $c = AB$, $x = AM$, $y = MB$, $h = CM$. Then we have $x : h = b : a$, $y : h = a : b$, so

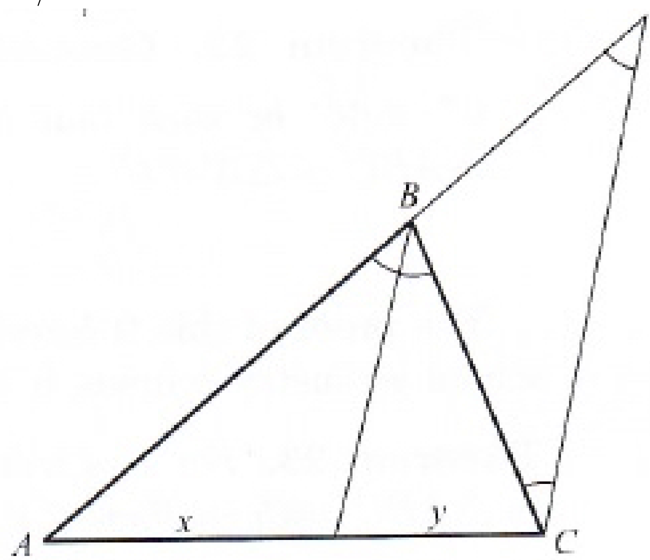
$$\frac{x}{h} \times \frac{y}{h} = 1$$

or $xy = h^2$.

## *Homework*

In the problems about constructing something with a ruler and compass, the ruler can only be used for drawing straight lines through two given points; you can not use it to measure distances. As before, you can freely use previous results and constructions instead of repeating all the steps.

1. Let $ABCD$ be a trapezoid with bases $AD = 9$, $BC = 6$, such that the height (distance between the bases) is equal to 5. Let $O$ be the intersection point of lines $AB$, $CD$.
   a. Show that triangles $\triangle OBC$, $\triangle OAD$ are similar and find the coefficient.
   b. Find the distance from $O$ to $AD$ (i.e., length of the perpendicular).
2. Given three line segments, of lenghts $1, x, y$, construct the line segments of lenghts $xy$, $x/y$, using only ruler and compass.
3. In a triangle $\triangle ABC$, let $D$ be midpoint of side $BC$, $E$ – midpoint of side $AC$, $F$ – midpoint of side $AB$. Prove that $\triangle DEF$ is similar to triangle $\triangle ABC$ with coefficient 1/2.
4. Use the folowing figure to prove that an angle bisector in a triangle $\triangle ABC$ divides the opposite side in the same proportion as the two adjoining sides: $\frac{x}{y} = \frac{BA}{BC}$.




5. In triangle $PQR$, $PQ$, $PR$ are the same. Point $S$ lies on $QR$ so that $QS = PS$ and $< RPS = 75^o$. What is the size of $< QRP$? (Math Kangaroo)