Math 7: Handout 24
Euclidean Geometry – 3.

1 *Congruence*

In general, two figures are called congruent if they have same shape and size.

- For line segments, it means that they have the same length: $\overline{AB} \cong \overline{CD}$ is the same as $AB = CD$.
- For angles, it means that they have the same measure: $\angle A \cong \angle B$ is the same as $m\angle A = m\angle B$.
- For triangles, it means that the corresponding sides are equal and corresponding angles are equal: $\triangle ABC \cong \triangle A'B'C'$ is the same as
$AB = A'B'$, $BC = B'C'$, $AC = A'C'$, $m\angle A = m\angle A'$, $m\angle B = m\angle B'$, $m\angle C = m\angle C'$.

Note that for triangles, the notation $\triangle ABC \cong \triangle A'B'C'$ not only tells that these two triangles are congruent, but also shows which vertex of the first triangle corresponds to which vertex of the second one. For example, $\triangle ABC \cong \triangle PQR$ is not the same as $\triangle ABC \cong \triangle QPR$.

2 *Congruence tests for triangles*

By definition, to check that two two triangles are congruent, we need to check that corresponding angles are equal and corresponding sides are equal; thus, we need to check 6 equalitites. However, it turns out that in fact, we can do with fewer checks.

Angle-Side-Angle Congruence Axiom (ASA) If $m\angle A = m\angle A'$, $m\angle B = m\angle B'$ and $AB = A'B'$, then $\triangle ABC \cong \triangle A'B'C'$.

Congruence Axiom (SSS) If $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$ then $\triangle ABC \cong \triangle A'B'C'$.

Congruence Axiom (SAS) If $AB = A'B'$, $AC = A'C'$ and $m\angle A = m\angle A'$, then $\triangle ABC \cong \triangle A'B'C'$.

3 *Isosceles triangles*

A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

Theorem If $\triangle ABC$ is isosceles, with base AC, then $m\angle A = m\angle C$.

Conversely, if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

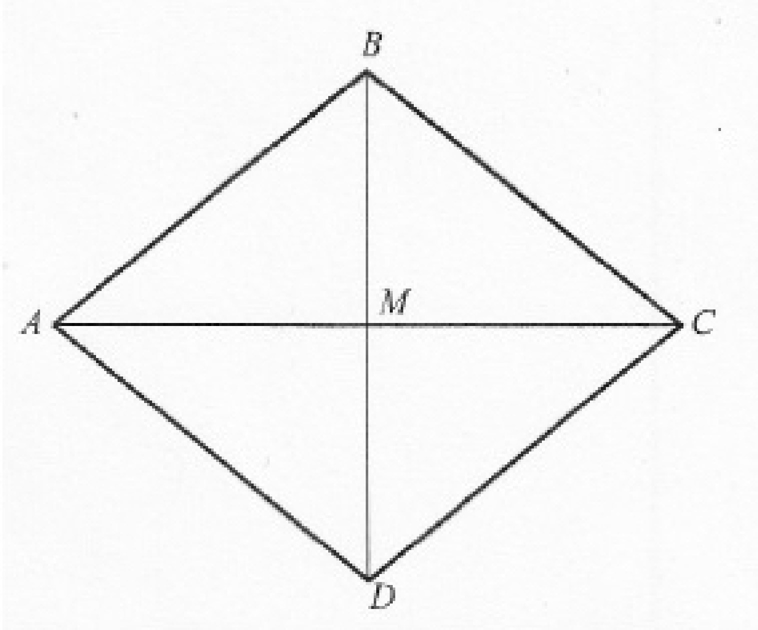
In any triangle, there are three special lines from each vertex. In $\triangle ABC$, the altitude from A is perpendicular to BC (it exists and is unique by Theorem 7); the median from A bisects BC (that is, it crosses BC at a point D which is the midpoint of BC); and the angle bisector bisects $\angle A$ (that is, if E is the point where the angle bisector meets BC, then $m\angle BAE = m\angle EAC$).

For general triangle, all three lines are different. However, it turns out that in an isosceles triangle, they coincide.

Theorem If B is the apex of the isosceles triangle ABC, and BM is the median, then BM is also the altitude, and is also the angle bisector, from B.

Homework

1. Let $\triangle ABC$ be such that all sides have equal length. Prove that then $m\angle A = m\angle B = m\angle C = 60^\circ$. [Such a triangle is called equilateral.]
2. Prove that if $\triangle ABC$ has $m\angle A = m\angle C$, then it is isosceles, with base AC.

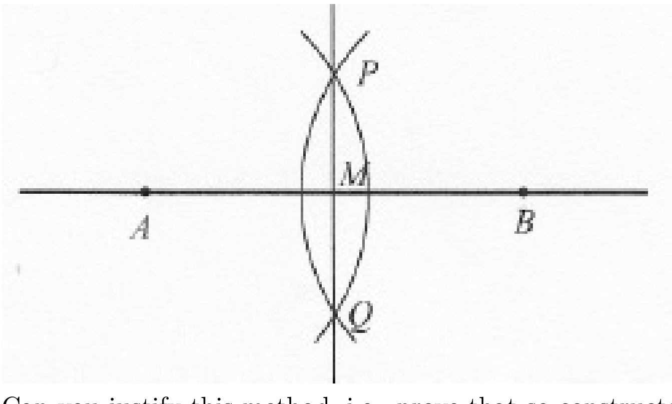


3. Let $ABCD$ be a quadrilateral such that $AB = BC = CD = AD$ (such a quadilateral is called rhombus). Let M be the intersection point of AC and BD.

a. Show that $\triangle ABC \cong \triangle ADC$
b. Show that $\triangle AMB \cong \triangle AMD$
c. Show that the diagonals are perpendicular and that the point M is the midpoint of each of the diagonals.

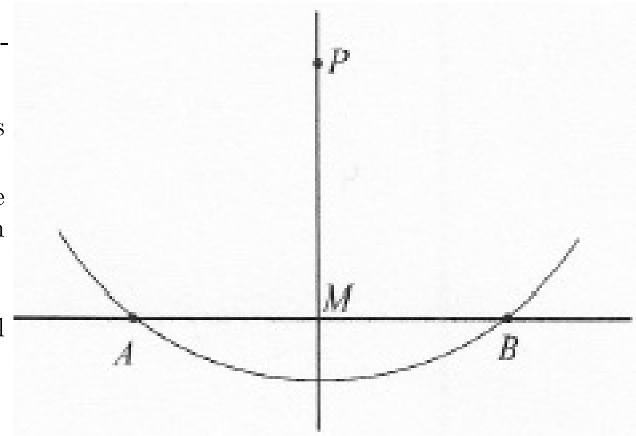
4. The following method explains how one can find the midpoint of a segment AB using a ruler and compass:

- Choose radius r (it should be large enough) and draw circles of radius r with centers at A and B.
- Denote the intersection points of these circles by P and Q. Draw a line $\overleftrightarrow{PQ}$.
- Let M be the intersection point of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{AB}$. Then M is the midpoint of AB.



Can you justify this method, i.e., prove that so constructed point will indeed be the midpoint of AB? You can use the defining property of the circle: for a circle of radius r, the distance from any point on this circle to the center is exactly r.[Hint: use the previous problem]

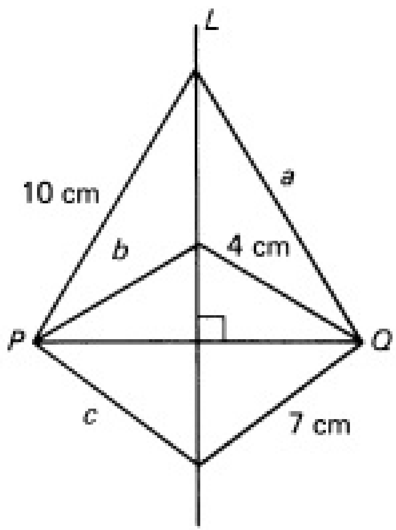
5. The following method explains how one can construct a perpendicular from a point P to line l using a ruler and compass:



- Choose radius r (it should be large enough) and draw circle of radius r with center at P.
- Let A, B be the intersection points of this circle with l. Find the midpoint M of AB (using the method of the previous problem). Then $\overleftrightarrow{MP} \perp l$.

   Can you justify this method, i.e., prove that so constructed $\overleftrightarrow{MP}$ will indeed be perpendicular to l?

6. In the triangle $\triangle ABC$, let M be a point on side AB. Prove that $m\angle BMC > m\angle A$.
7. What is the angle between the hour hand and minute hand of a clock at 11:10 AM? (Mathcounts)
8. Line L is the perpendicular bisector of PQ. Find a, b and c.



9. Line K is the perpendicular bisector of PQ, line L is the perpendicular bisector of QM. Show that $OP = OM$.