Math 7: Handout 23
Euclidean Geometry – 2.

1 *Parallel and perpendicular lines*

Theorem 6. *Given a line l and point P not on l, there exists exactly one line m through P which is parallel to l.*

Proof. **Existence:** Let us draw a line k through P which intersects l. Now draw a line m through P such that alternate interior angles are equal: $m\angle 1 = m\angle 2$. Then, by Axiom 4 (alternate interior angles), we have $m \parallel l$.
Uniqueness: To show that such a line is unique, let us assume that there are two different lines, m_1, m_2 through P both parallel to l. By Theorem 2, this would imply $m_1 \parallel m_2$. This gives a contradiction, because they both go through P(Figure 1).

□

Theorem 7. *Given a line l and a point P not on l, there exists a unique line m through P which is perpendicular to l.*

2 *Sum of angles of a triangle*

Definition 1. A triangle is a figure consisting of three distinct points A, B, C (called vertices) and line segments $\overline{AB}, \overline{BC}, \overline{AC}$. We denote such a triangle by $\triangle ABC$.

Similarly, a quadrilateral is a figure consisting of 4 distinct points A, B, C, D and line segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ such that these segments do not intersect except at A, B, C, D.

Theorem 8. *The sum of measures of angles of a triangle is* $180°$.

Proof. Draw a line m through B parallel to $\overleftrightarrow{AC}$ (possible by Theorem 6). Let D, E be points on m as shown in the Figure 2.

Then $m\angle DBA = m\angle A$ as alternate interior angles, $m\angle CBE = m\angle C$. On the other hand, by Axiom 3 (angles add up), we have

$$m\angle DBA + m\angle B + m\angle CBE = 180°$$

Thus, $m\angle A + m\angle B + m\angle C = 180°$. □

Theorem 9. *For a triangle $\triangle ABC$, let D be a point on continuation of side AC, so that C is between A and D. Then $m\angle CBD = m\angle A + m\angle B$. (Such an angle is called the exterior angle of triangle ABC.)*

Theorem 10. *Sum of angles of a quadrilateral is equal to* $360°$.

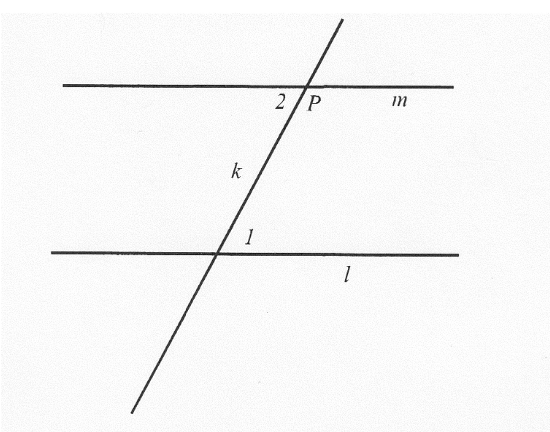


Figure 1. Parallel Lines

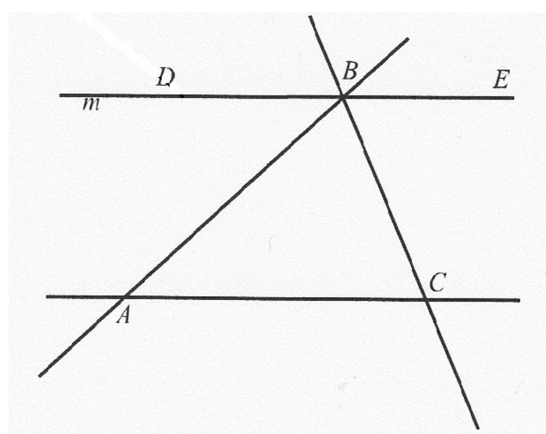
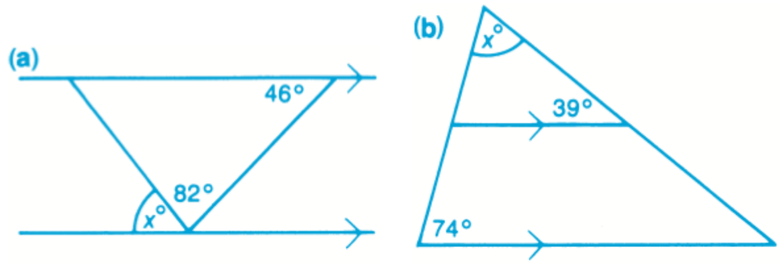


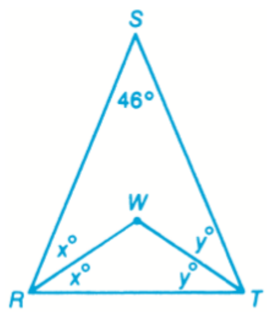
Figure 2. Sum of Angles

3 *Homework*

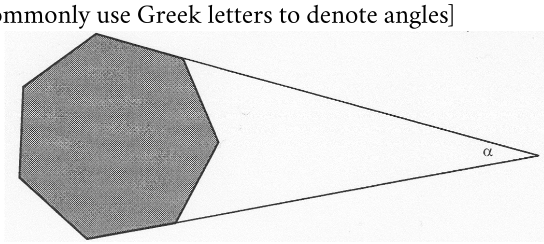
1. In each of the following pictures find the value of x:



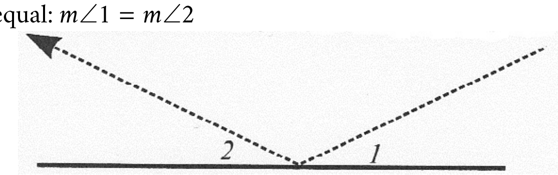
2. Find the measure of angle $\angle RWT$:



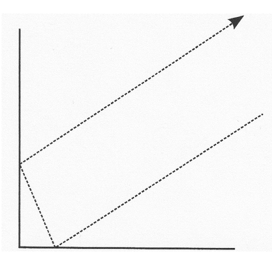
3. Prove Theorem 7.
4. Prove Theorem 9.
5. Deduce a formula for the sum of angles in a polygon with n vertices.
6. In the figure below, all angles of the 7-gon are equal. What is angle α?[By the way: α is a Greek letter, pronounced "alpha"; mathematicians commonly use Greek letters to denote angles]



7. The reflection law states that the angles formed by the incoming light ray and the reflected one with the surface of the mirror are equal: $m\angle 1 = m\angle 2$



   Using this law, show that a corner made of two perpendicular mirrors will reflect any light ray exactly back: the reflected ray is parallel to the incoming one:



   This property – or rather, similar property of corners in 3-D – is widely used: reflecting road signs, tail lights of a car, reflecting strips on clothing are all contrsructed out of many small reflecting corners so that they reflect the light of a car headlamp exactly back to the car.

8. Show that if, in a quadrilateral $ABCD$, diagonally opposite angles are equal ($m\angle A = m\angle C$, $m\angle B = m\angle D$), then opposite sides are parallel. [Hint: show first that $m\angle A + m\angle B = 180°$.]