Math 7: Handout 22
Euclidean Geometry – 1.

Today we started the study of Euclidean geometry. Euclidean geometry tries to describe geometric properties of various figures in the plane. Figures are understood as sets of points; we will use capital letters for points and write $P \in m$ for "point P lies in figure m", or "figure m contains point P". The notion of "point" can not be defined: it is so basic that it is impossible to explain it in terms of simpler notions. In addition, there are some other basic notions (lines, distances, angles) that can not be defined. Instead, we can state some basic properties of these objects; these basic properties are usually called "postulates" or "axioms of Euclidean geometry". **All results in Euclidean geometry should be proved by deducing them from the axioms**; justifications "it is obvious", "it is well-known", or "it is clear from the figure" are not acceptable. We allow use of all logical rules. I assume that you are familiar with some basic logical reasoning, in particular with indirect proof (also known as proof by contradiction): if assumption A leads to a contradiction, it means that A must be false. We will also use all the usual properties of numbers, equations, inequalities, etc.

1 *Basic objects*

These objects are the basis of all our constructions: all objects we will be discussing will be defined in terms of these objects. No definition is given for these basic objects.

- Points
- Lines
- Distances: for any two points A, B, there is a non-negative number AB, called distance between A, B.
- Angle measures: for any angle $\angle ABC$, there is a real number $m\angle ABC$, called the measure of this angle (more on this later).

We will also frequently use words "between" when describing relative position of points on a line (as in: A is between B and C) and "inside" (as in: point C is inside angle $\angle AOB$).

Having these basic notions, we can now define more objects. Namely, we can give definitions of

- an interval, or line segment is a part of a line consisting of two points, called end points, and the set of all points between them. ($\overline{AB}$)
- a ray is a part of a line consisting of a given point, called the end point, and the set of all points on one side of the end point.
- an angle is the union of two rays having the same end point. The end point is called the vertex of the angle, and the rays are called the sides of the angle. (notation: $\angle AOD$)
- parallel lines: two distinct lines l, m are called parallel (notation: $l \parallel m$) if they do not intersect, i.e. have no common points

2 *First postulates*

Axiom 1. *For any two distinct points A, B, there is a unique line containing these points (this line is usually denoted $\overleftrightarrow{AB}$).*

Axiom 2. *If points A, B, C are on the same line, and B is between A and C, then $AC = AB + BC$*

Axiom 3. *If point B is inside angle $\angle AOC$, then $m\angle AOC = m\angle AOB + m\angle BOC$. Also, the measure of a straight angle is equal to $180°$. (see Figure 1)*

Axiom 4. *Let line l intersect lines m, n and angles $\angle 1$, $\angle 2$ are as shown in Figure 2 below (in this situation, such a pair of angles is called alternate interior angles). Then $m \parallel n$ if and only if $m\angle 1 = m\angle 2$.*

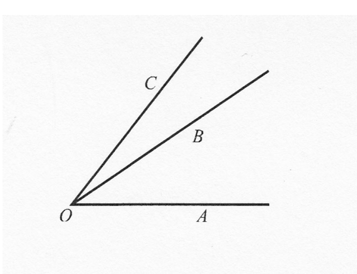


Figure 1. Angle Addition

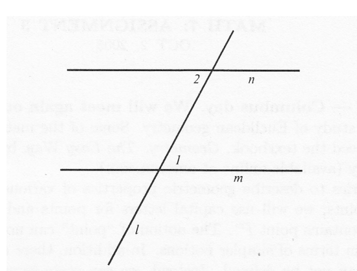


Figure 2. Alt. Int. Angles

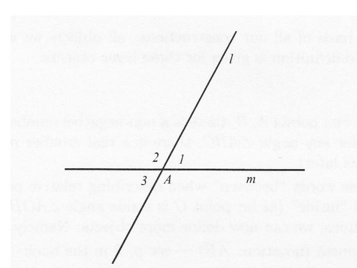


Figure 3. Vertical Angles

3 *First theorems*

Theorem 1. *If lines l, m intersect, than they intersect at exactly one point.*

Proof. Assume that they intersect at more than one point. Let P, Q be two of the points where they intersect. Then both l, m go through P, Q. This contradicts Axiom 1. Thus, our assumption (that l, m intersect at more then one point) must be false. □

Theorem 2. *If $l \parallel m$ and $m \parallel n$, then $l \parallel n$*

Theorem 3. *Let A be the intersection point of lines l, m, and let angles $1, 3$ be as shown in the figure below (such a pair of angles are called vertical). Then $m\angle 1 = m\angle 3$.*

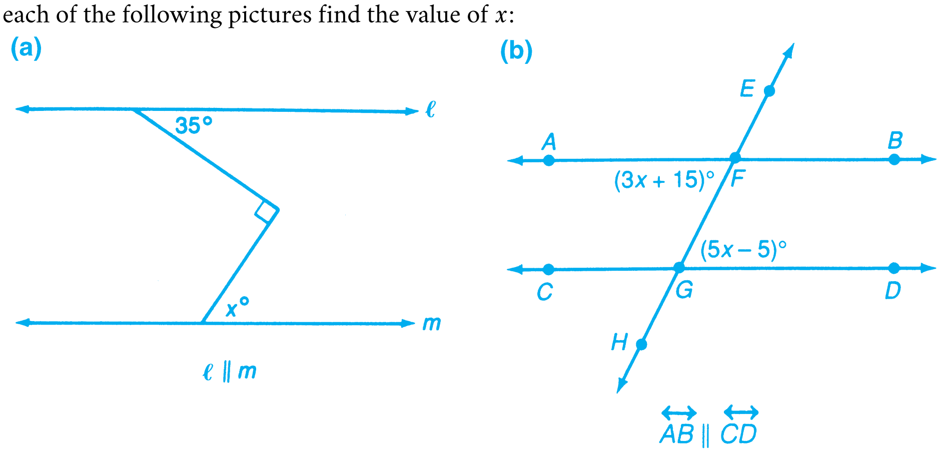
Proof. Let angle 2 be as shown in the Figure 3 . Then, by Axiom 3, $m\angle 1 + m\angle 2 = 180°$, so $m\angle 1 = 180° - m\angle 2$. Similarly, $m\angle 3 = 180° - m\angle 2$. Thus, $m\angle 1 = m\angle 3$. □

Theorem 4. *Let l, m be intersecting lines such that one of the four angles formed by their intersection is equal to $90°$. Then the three other angles are also equal to $90°$. (In this case, we say that lines l, m are* perpendicular *and write $l \perp m$.)*

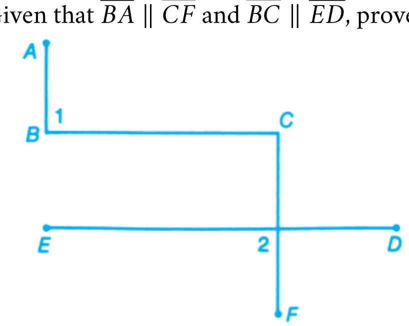
Theorem 5. *Let l_1, l_2 be perpendicular to m. Then $l_1 \parallel l_2$.*
Conversely, if $l_1 \perp m$ and $l_2 \parallel l_1$, then $l_2 \perp m$.

4 *Homework*

1. Prove Theorem 2. [*Hint:* assume that l and n are not parallel; then they must intersect at some point P…]
2. Prove Theorem 4.
3. Prove Theorem 5.
4. Suppose that instead of studying geometry on the plane, we study geometry on the sphere (say, Earth surface) and take lines to be equators, i.e. intersections of the sphere with a plane going through the center of the sphere. Which of the axioms will be true in this new, "spherical", geometry? Which will be false? Can you suggest a new set of axioms to describe this geometry?
5. In each of the following pictures find the value of x:



6. Given that $\overline{BA} \parallel \overline{CF}$ and $\overline{BC} \parallel \overline{ED}$, prove that $m\angle 1 = m\angle 2$.



7. Suppose we draw k lines on the plane so that each of them intersects each other, and all intersection points are distinct. Into how many pieces will they cut the plane? [Hint: how does the number of pieces change when you increase k by 1, i.e. add one more line?]