Summary of Quadratic Equations. Properties of Parabola.

Summary

- A **quadratic polynomial** is an expression of the form $p(x) = ax^2 + bx + c$.

- **Roots** of a quadratic polynomial are numbers such that $p(x) = 0$. If x_1, x_2 are roots, then $p(x) = a(x - x_1)(x - x_2)$.

- **Vietá formulas**: If x_1, x_2 are roots of $x^2 + bx + c$, then

$$x_1 + x_2 = -b \tag{1}$$
$$x_1 x_2 = c \tag{2}$$

- **Completing the square**: we can rewrite

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a} = a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a^2}\right) \tag{3}$$

where $D = b^2 - 4ac$.

From this, one gets the **quadratic formula**: if $D < 0$, there are no roots; if $D \geq 0$, then the roots are

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \tag{4}$$

- From formula (3), we see that:

  - If $a > 0$, then the **smallest** possible value of $p(x)$ is $-\frac{D}{4a}$, which happens when $x = -\frac{b}{2a}$. In this case the graph is a parabola with branches going up.
  - If $a < 0$, then the **largest** possible value of $p(x)$ is $-\frac{D}{4a}$, which happens when $x = -\frac{b}{2a}$. In this case the graph is a parabola with branches going down.

Properties of a Parabola

A parabola is the set of all points in a plane that are equally distant away from a given point and a given line (see black dotted lines).

This given point is called the **focus** (black dot) of the parabola and the line is called the **directrix** (green line).

If the parabola is of the form $(x - h)^2 = 4p(y - k)$, the vertex is $(h.k)$, the focus is $(h, k + p)$ and directrix is $y = k - p$
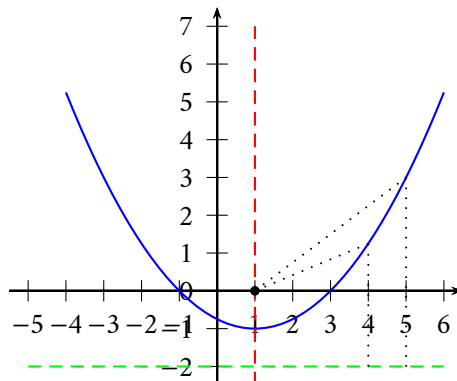
Homework

1. For what values of a does the polynomial $x^2 + ax + 14$ has no roots? exactly one root? two roots?

2. Let x_1, x_2 be the roots of the equation $x^2 + 3x + 4 = 0$. Without calculating the roots, find:
   a. $x_1^2 + x_2^2$
   b. $\frac{1}{x_1^2} + \frac{1}{x_2^2}$

3. A circle with center $(3, 5)$ intersects the y-axis at $(0, 1)$.

   - Find the radius of the circle
   - Find the coordinates of the other point of intersection on the y-axis
   - What are the coordinates of the intersection points of the circle with the x-axis?

4. Of all the rectangles with perimeter 4, which one has the largest area?
   [**Hint:** if sides of the rectangle are a and b, then the area is $A = ab$, and the perimeter is $2a + 2b = 4$. Thus, $b = 2 - a$, so one can write A using only a...]

5. Prove that for any point P on the parabola $y = \frac{x^2}{4} + 1$, the distance from P to the x-axis is equal to the distance from P to the point $(0, 2)$.

6. Graph $x^2 = 4y$. What is the focus, directrix and vertex of the parabola?

7. Graph $y = (\sqrt{x})^2$. Note $x \geq 0$

8. A triangle ABC, has corners $A(-3, 0)$, $B(0, 3)$ and $(3, 0)$. The line $y = \frac{1}{3}x + 1$ separates the triangle in 2. What is the area of the piece lying below the line?

*9. Find all intersection points of parabola $y = x^2$ and the circle with radius $\sqrt{6}$ and center at $(0, 4)$v.