

Math 7: Handout 10

Binomial Coefficients.

Binomial Coefficients

The k -th entry in n -th line of a Pascal Triangle is denoted by $\binom{n}{k}$, or by $\binom{n}{k}$. Note that both n and k are counted from 0, not from 1: for example, $\binom{6}{2} = 15$.

We saw that these numbers appear in many problems:

$$\begin{aligned}\binom{n}{k} &= \text{The number of paths on the chessboard going } k \text{ units up and } n - k \text{ to the right} \\ &= \text{The number of words that can be written using } k \text{ zeros and } n - k \text{ ones} \\ &= \text{The number of ways to choose } k \text{ items out of } n \text{ (order doesn't matter)}\end{aligned}$$

It turns out that there is an explicit formula for $\binom{n}{k}$:

$$\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}$$

Compare it with the number of ways of choosing k items out of n when the order matters:

$${}_nP_k = n(n-1)\dots(n-k+1) = \frac{n!}{(n-k)!}$$

For example, there are $5 \cdot 4 = 20$ ways to choose 2 items out of 5 if the order matters, and $\frac{5 \cdot 4}{2} = 10$ if the order doesn't matter.

Homework

1. If we draw 3 cards out of the deck, what are the chances that
 - They will all be all spades
 - They will be all aces
 - That they will be ace of spades, queen of spades, and king of spades, in this order
 - That they will be queen of spades, ace of spades, and king of spades, in this order
 - That they will be ace, queen, and king of spades, in some order
2. Are there any rows in the Pascal triangle where all numbers are odd? Which rows are they?
3. How many words one can get by permuting letters of the word “tiger”? of the word “rabbit”? of the word “mammoth”?
4. In one of the lotteries run by New York State, “Sweet Million”, they randomly choose 6 numbers out of numbers 1–40. If you guess all 6 correctly (order does not matter), you win \$1,000,000. [There are also smaller prizes for guessing 5 out of 6, etc., but let us ignore them for now.]
 - a. How many ways are there to choose 6 numbers out of 40?
 - b. What are your chances of winning?
 - c. If a lottery ticket cost \$1, how much money does New York State make for each ticket sold (on average)?

Bonus question: find online the rules for another NY lottery, “Mega Millions”, and analyze your chances to win.
5. In poker, players are drawing “hands” (combinations of 5 cards) from the 52-card deck (4 suits, 13 cards in each).
 - a. How many possible hands are there?
 - b. What are your chances of drawing a hand in which all cards are spades?
 - c. What are your chances of drawing a hand which has 4 queens in it? [Hint: how many such hands are there?]
 - d. What are your chances of drawing a royal flush (Ace, King, Queen, Jack, 10 — all of the same suit)? [Hint: what are your chances of drawing a royal flush in a given suit, say spades?]
- *6.. Prove the formula for $\binom{n}{k}$ given above. [Hint: since we had already proved the formula for ${}_nP_k$, it would be enough to prove that $\binom{n}{k} = {}_nP_k/k!$, or ${}_nP_k = \binom{n}{k} \cdot k!$. Think about the example of choosing 3 students out of 10...]