MATH 7
HOMEWORK 28: FIBONACCI NUMBERS AND THE GOLDEN RATIO

Definition 1. The Fibonacci numbers is the sequence of numbers constructed by the following rule: $F_1 = F_2 = 1$ and for $n > 1$, $F_{n+1} = F_n + F_{n-1}$. Here are the first several Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, ...
(Some people also define $F_0 = 0$.)

Homework

1. [attributed to certain Leonardo of Pisa, also called Fibonacci, 1202]

Somebody buys a pair of rabbits and places them in a pen. The nature of rabbits is such that each month a pair of rabbits gives birth to another pair, and they start reproducing upon reaching the age of 2 months.
How many pairs of rabbits will he have in one year (considering the rabbits immortal)?

month		number of pairs
1		1
2		1
3		2
4		3
5		5

2. Show that for any n, $F_1 + F_2 + \dots F_n = F_{n+2} - 1$. [Hint: $F_{n+2} = F_{n+1} + F_n$.]

3. (a) Which of Fibonacci numbers are even? Find the pattern and try to explain why this pattern holds.
(b) Which of Fibonacci numbers are divisible by 3? Find the pattern and try to explain why this pattern holds.

4. Let

$$\Phi = \frac{1+\sqrt{5}}{2} \approx 1.618\dots$$
$$\bar{\Phi} = \frac{1-\sqrt{5}}{2} \approx -0.618\dots$$

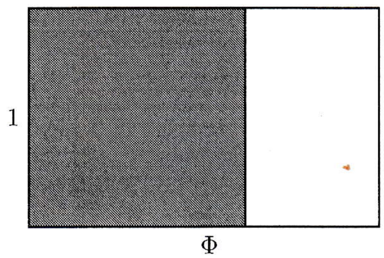
(a) Show that $\Phi, \bar{\Phi}$ satisfy equation $x^2 = x + 1$
(b) Show that the geometric progression $a_1 = 1, a_2 = \Phi, a_3 = \Phi^2, \dots$ satisfies the same rule as the Fibonacci sequence:

$$a_{n+2} = a_n + a_{n+1}$$

[Hint: do not use explicit formulas for $\Phi, \bar{\Phi}$. Instead, use that $\Phi^{n+2} = \Phi^{n+1} + \Phi^n$, and similarly for $\bar{\Phi}$.]

The number Φ appears in many places in mathematics. It is called the *Golden Ratio* (in old times, it was also sometimes called the *Divine proportion*). There are whole books about it. By the way, Φ is the Greek letter which reads "phi".

5. Consider the rectangle with sides 1 and Φ. Show that if we cut from it a 1×1 square, then the remaining rectangle will again have proportions $1 : \Phi$.



6. (a) Continue the following sequence:
$0, 1, 3, 6, 10, 15, \ldots$
[These are so-called triangular numbers — if you arrange n circles in the triangle, so that the top layer has 1 circle, the next, two circles, etc]

(b) Continue the following sequence
$0, 1, 4, 10, 20, 35, \ldots$
These are called the tetrahedral numbers — number of balls in a tetrahedral pyramid

***7.** Do you think the infinite sum

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots +$$

makes sense? if so, what is it equal to?

***8.** Let us put some weights in each cell of (infinite) sheet of square ruled paper as shown in the figure below:

$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Assuming that the pattern repeats infinitely far to the right and up, what will be the total weight of all the squares?