# MATH 7: HOMEWORK 22

MARCH 17, 2019

## 1. Arcs

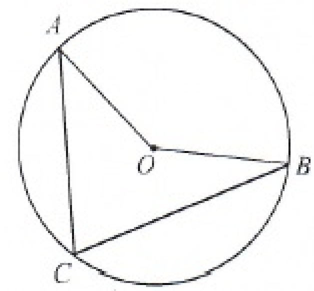
Let $A, B$ be two distinct points on a circle $\Sigma$. An arc $\overset{\frown}{AB}$ is the part of the circle bounded by these two points; more formally, it can be defined as the intersection of angle $\angle AOB$ with $\Sigma$ (here $O$ is the center of the circle; angles with vertex at $O$ are called central angles). The angle measure of an arc is defined to be the angle measure of the corresponding central angle:

$$m\ \overset{\frown}{AB} = m\angle AOB$$

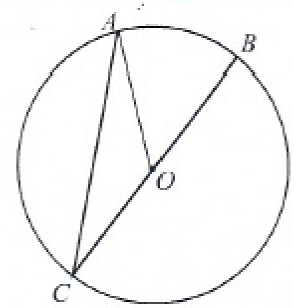
Note that the notation $\overset{\frown}{AB}$ is ambiguous: there are two arcs bounded by $A, B$ (just as there are two angles bounded by rays $\overrightarrow{OA}, \overrightarrow{OB}$).

## 2. Central angle theorem

Let $A, B, C$ be three points on a circle $\Sigma$. It turns out that there is a simple relation between the measure of $\angle ACB$ and the measure of the arc $\overset{\frown}{AB}$.




**Theorem 1** (Central angle theorem)**.** *Let $\Sigma$ be a circle with center $O$, and let $A, B, C$ be distinct points on $\Sigma$. Then $m\angle ACB = \frac{1}{2} m\angle AOB$.*




*Proof.* We will first prove a special case of the theorem, and later show how the general statement can be deduced from this special case.
Consider the special case when $CB$ is a diameter, i.e. goes through the center $O$ of the circle. Since $AO = OC$, $\triangle AOC$ is isosceles and thus $\angle A \cong \angle C$. Since $\angle AOB$ is the exterior angle of $\triangle AOC$, we have $m\angle AOB = m\angle A + m\angle C = 2m\angle C$. This completes the proof of the special case.
The proof of the general case was discussed in class. □

This theorem can be reversed.

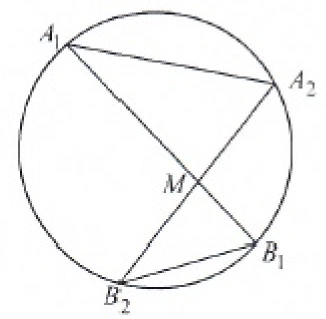
**Theorem 2** (Central angle theorem 2)**.** *Let $\Sigma$ be a circle with center $O$, and let $A, B$ be distinct points on $\Sigma$. Let $C$ be such that $m\angle ACB = \frac{1}{2} m\angle AOB$. Then $C$ also lies on the circle $\Sigma$.*

These theorems have a number of corollaries. Here are some.

**Theorem 3.** *Let $A, B, C$ be distinct points. Then $\angle ACB$ is a right angle if and only if $C$ lies on the circle with diameter $AB$.*

**Theorem 4.** *A quadrialateral $ABCD$ can be inscribed in a circle if and only if sums of opposite angles are equal to $180°$: $m\angle A + m\angle C = m\angle B + m\angle D = 180°$.*

**Theorem 5.** *Let $M$ be a point inside the circle $\Sigma$. Then for any chord*[1] *$AB$ passing through $M$, the product $AM \cdot MB$ is the same.*




*Proof.* Let $A_1B_1, A_2B_2$ be two such chords. Consider the triangles $\triangle A_1A_2M$, $\triangle B_2B_1M$. By central angle theorem,

$$m\angle A_1 = \frac{1}{2} m \overset{\frown}{A_2B_1} = m\angle B_2$$

similarly, $m\angle A_2 = m\angle B_1$. Therefore, these two triangles are similar by AA: $\triangle A_1A_2M \sim \triangle B_2B_1M$. Therefore,

$$\frac{A_1M}{B_2M} = \frac{A_2M}{B_1M}$$

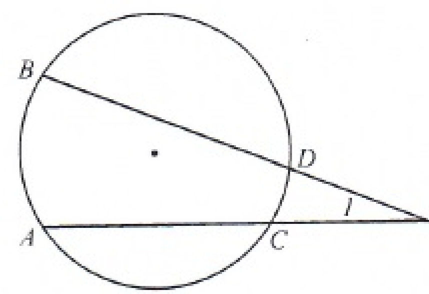
Cross-multiplying, we get $A_1M \cdot B_1M = A_2M \cdot B_2M$. Thus, for any two such chords, the product is the same. $\square$

## Homework

1. Do problems 1-4 on page 307 in the geometry textbook.
2. Let $ABCD$ be a trapezoid with bases $AD$, $BC$ and let $M, N$ be midpoints of $AD$, $BC$. Prove that line $\overleftrightarrow{MN}$ passes through the intersection point $P$ of lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$. [Hint: can you prove that line $PM$ goes through $N$?]
3. Let $C, B$ be points on the circle $\Sigma$ and let $A$ be such that $AC$ is the tangent line to $\Sigma$. Prove that then $m\angle ACB = \frac{1}{2} m \overset{\frown}{BC}$.
4. Let $M$ be a point outside the circle $\Sigma$.
   (a) Let $l$ be a line through $M$ which intersects $\Sigma$ at points $A, B$. Show that then the product $MA \cdot MB$ does not depend on the choice of $l$. [Hint: let $l_1, l_2$ be two such lines and $A_1, B_1, A_2, B_2$ be corresponding intersection points. Arguing similar to the proof of Theorem 5, show that then $MA_1 \cdot MB_1 = MA_2 \cdot MB_2$.]
   (b) Show that the product $MA \cdot MB$, discussed in the previous part, is equal to $MC^2$, where $C$ is the point on $\Sigma$ such that $MC$ is the tangent line.
5. Prove Theorem 4
6. Given segments of length $x, y$, construct a segment of length $\sqrt{xy}$, using only ruler and compass. [Hint: use Theorem 3 and similar triangles created when using the altitude in a right triangle in homework 18, last theorem on the first page.]
7. Given a circle $\Sigma$ with center $O$ and a point $P$ outside it, construct a tangent line to $\Sigma$ through $P$ using ruler and compass. [The problem has many solutions. Here is one of them: if $A$ is the tangency point, then $PA \perp OA$; now we can use Theorem 3.]
8. Consider an angle such that its vertex is outside the circle $\Sigma$, and each of its sides intersects $\Sigma$ at two points. Prove that then the measure of this agle is equal to half of the difference of the two intercepted arcs:

$$m\angle 1 = \frac{1}{2}(m \overset{\frown}{AB} - m \overset{\frown}{CD})$$

[Hint: draw segment $AD$ and use exterior angle theorem]




[1]Recall that a chord is a segment both ends of which lie on the circle