MATH 7: HOMEWORK 19

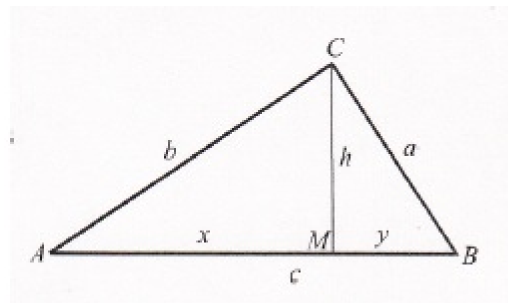
FEBRUARY 24, 2019

1. Pythagorean Theorem

The following theorem is one of the oldest known to humanity. According to some studies, this result may have been known as early as 4 thousand years ago — long before Pythagoras or Euclid. There is a number of ways to prove this theorem. The proof given below is not the most geometrically intuitive — but it is the easiest one to derive from the axioms.

Theorem 1. *Let $\triangle ABC$ be a right triangle, with $\angle C$ being the right angle. Denote $a = BC$, $b = AC$, $c = AB$. Then*

$$a^2 + b^2 = c^2$$



Proof. Let CM be the altitude from vertex C. Triangles $\triangle ABC$, $\triangle ACM$, $\triangle CBM$ are similar; thus, $\frac{x}{b} = \frac{b}{c}$, so $x = b^2/c$. Similarly, $y = a^2/c$. Therefore,

$$c = x + y = \frac{a^2}{c} + \frac{b^2}{c}$$

Multiplying both sides by c, we get the Pythagorean theorem. □

This theorem can be reversed:

Theorem 2. *If $AB^2 = AC^2 + BC^2$, then $\triangle ABC$ is a right triangle, with right angle C.*

Proof. Construct a right triangle $\triangle A'B'C'$, with $m\angle C' = 90^\circ$ and $C'B' = CB$, $C'A' = CA$. Then, by Pythagorean theorem, $A'B' = \sqrt{AC^2 + AB^2} = AB$; thus, $\triangle A'B'C' \cong \triangle ABC$ by SSS. Therefore, $m\angle C = m\angle C' = 90^\circ$. □

2. Right Triangles Congruences

Theorem 3 (Congruence by HL)**.** *If two right triangles have congruent hypothenuses and a congruent pair of legs: $c = c'$, $a = a'$, then these triangles are congruent. (similar SSS)*

Theorem 4 (Congruence by LL)**.** *If two right triangles have both equal pair of legs: $b = b'$, $a = a'$, then these triangles are congruent. (similar SAS)*

Theorem 5 (Congruence by LA)**.** *If two right triangles have a corresponding congruent pair of acute angles and a corresponding congruent pair of legs, then these triangles are congruent. (similar ASA)*

Theorem 6 (Congruence by HA)**.** *If two right triangles have a congruent pair of acute angles and a congruent pair of hypotenuse, then these triangles are congruent. (similar ASA)*

3. Perpendicular bisectors and angle bisectors in a triangle

A perpendicular bisector of a segment AB is a line l which is perpendicular to AB and contains the midpoint M of AB.

Theorem 7. *For given points A, B, a point P is equidistant from A, B if and only if P is on the perpendicular bisector l of AB: $AP = BP \iff P \in l$.*

This theorem is sometimes formulated by saying that the perpendicular bisector is the *locus* of points equidistant from A, B.

There is a similar result for points equidistant from two given lines. Namely, define the distance from a point P to the line l to be the length of perpendicular PM from P to l (by one of your previous homework problems, this is the shortest distance from P to l).

Theorem 8. *For given intersecting lines l, m, a point P is equidistant from l, m if and only if P is on an angle bisector of one of the four angles formed by l, m.*

Theorem 9. *In any triangle $\triangle ABC$, the three perpendicular bisectors of the sides of the triangle interesect at a single point P. This point is equidistant from all three vertices: $AP = BP = CP$.*

Proof. Let l be the perpendicular bisector of side AB, and m the perpendicular bisector of side BC. Let P be the intersection point of l, m. Then $AP = BP$ (by Theorem 7, since $P \in l$); similarly, $BP = CP$. Therefore, $AP = CP$; thus, by Theorem 7, P lies on the perpendicular bisector of side AC.

$\square$

Theorem 10. *In any triangle $\triangle ABC$, the three angle bisectors interesect at a single point Q. This point is equidistant from all three sides of the triangle.*

Homework

1. Solve problems 29,30,31,32 on page 231 in the E-Z Geometry book.

2. Prove Theorems $3, 4, 5, 6$.

3. Prove Theorem 8.

4. Prove Theorem 10.

5. In a trapezoid $ABCD$, with bases AD, BC, it is given that $AD = 13$, $BC = 7$, and the distance between bases is 4. It is also given that $AB = CD$. Find AB, AC.