## MATH 7: HOMEWORK 18

FEB 10, 2019

## 1. Similar triangles

We say that triangles $\triangle A B C$ and $\triangle A^{\prime} B^{\prime} C^{\prime}$ are similar with coefficient $k$ if $m \angle A=m \angle A^{\prime}, m \angle B=m \angle B^{\prime}$, $\angle C=\angle C^{\prime}$ and

$$
\frac{A^{\prime} B^{\prime}}{A B}=\frac{B^{\prime} C^{\prime}}{B C}=\frac{A^{\prime} C^{\prime}}{A C}=k
$$

We will use notation $\triangle A B C \sim \triangle A^{\prime} B^{\prime} C^{\prime}$.

Theorem Consider a triangle $\triangle A B C$ and let $B^{\prime} \in \overrightarrow{A B}, C^{\prime} \in \overrightarrow{A C}$ be such that lines $\overleftrightarrow{B C}$ and $\overleftrightarrow{B^{\prime} C^{\prime}}$ are parallel. Then $\triangle A B C \sim$ $\triangle A^{\prime} B^{\prime} C^{\prime}$.


Theorem For any triangle $\triangle A B C$ and a real number $k>0$, there exists a triangle $\triangle A^{\prime} B^{\prime} C^{\prime}$ similar to $\triangle A B C$ with coefficient $k$.

Theorem [Similarity via AA] Let $\triangle A B C, \triangle A^{\prime} B^{\prime} C^{\prime}$ be such that $m \angle A=m \angle A^{\prime}, m \angle B=m \angle B^{\prime}$. Then these triangles are similar.

Proof Let $k=\frac{A^{\prime} B^{\prime}}{A B}$. Construct a triangle $\triangle A^{\prime \prime} B^{\prime \prime} C^{\prime \prime}$ which is similar to $\triangle A B C$ with coefficient $k$. Then $A^{\prime} B^{\prime}=A^{\prime \prime} B^{\prime \prime}$, and $m \angle A=m \angle A^{\prime}=m \angle A^{\prime \prime}, m \angle B=m \angle B^{\prime}=m \angle B^{\prime \prime}$. Thus, by ASA, $\triangle A^{\prime} B^{\prime} C^{\prime} \cong$ $\triangle A^{\prime \prime} B^{\prime \prime} C^{\prime \prime}$ 。

Theorem [Similarity via SAS] Let $\triangle A B C, \triangle A^{\prime} B^{\prime} C^{\prime}$ be such that $\angle A=\angle A^{\prime}, \frac{A^{\prime} B^{\prime}}{A B}=\frac{A^{\prime} C^{\prime}}{A C}$. Then these triangles are similar.

Theorem [Similarity via SSS] Let $\triangle A B C, \triangle A^{\prime} B^{\prime} C^{\prime}$ be such that

$$
\frac{A^{\prime} B^{\prime}}{A B}=\frac{A^{\prime} C^{\prime}}{A C}=\frac{B^{\prime} C^{\prime}}{B C}
$$

Then these triangles are similar.
One of the most important applications of the theory of similar triangles is to the study of right triangles and Pythagorean theorem.

A right triangle is a triangle in which one of the angles is a right angle. A hypotenuse is the side opposing the right angle; two other sides are called legs.

Theorem Let $\triangle A B C$ be a right triangle, with $\angle C$ being the right angle. Let $C M$ be the altitude of angle $C$. Then triangles $\triangle A B C, \triangle A C M$, $\triangle C B M$ are all similar.
Proof It immediately follows from AA similarity rule.


This theorem immediately implies a number of important relations between various lengths in these triangles. We will give one of them. Denote for brevity $a=B C, b=A C, c=A B, x=A M, y=M B$, $h=C M$. Then we have $x: h=b: a, y: h=a: b$, so

$$
\frac{x}{h} \times \frac{y}{h}=1
$$

or $x y=h^{2}$.

## Homework

In the problems about constructing something with a ruler and compass, the ruler can only be used for drawing straight lines through two given points; you can not use it to measure distances. As before, you can freely use previous results and constructions instead of repeating all the steps.

1. Problems $8,9,12,17,18,19,20$ starting on page 226 in the textbook.
2. Let $A B C D$ be a trapezoid with bases $A D=9, B C=6$, such that the height (distance between the bases) is equal to 5 . Let $O$ be the intersection point of lines $A B, C D$.
(a) Show that triangles $\triangle O B C, \triangle O A D$ are similar and find the coefficient.
(b) Find the distance from $O$ to $A D$ (i.e., length of the perpendicular).
3. Given three line segments, of lenghts $1, x, y$, construct the line segments of lenghts $x y, x / y$, using only ruler and compass.
4. In a triangle $\triangle A B C$, let $D$ be midpoint of side $B C, E$ - midpoint of side $A C, F$ - midpoint of side $A B$. Prove that $\triangle D E F$ is similar to triangle $\triangle A B C$ with coefficient $1 / 2$.
5. Use the folowing figure to prove that an angle bisector in a triangle $\triangle A B C$ divides the opposite side in the same proportion as the two adjoining sides: $\frac{x}{y}=\frac{B A}{B C}$.

6. In triangle $P Q R, P Q, P R$ are the same. Point $S$ lies on $Q R$ so that $Q S=P S$ and $<R P S=75^{\circ}$. What is the size of $<Q R P$ ? (Math Kangaroo)
