## MATH 7: HOMEWORK 17

FEBRUARY 3, 2019

## 1. Congruence

In general, two figures are called congruent if they have same shape and size.

- For line segments, it means that they have the same length: $\overline{A B} \cong \overline{C D}$ is the same as $A B=C D$.
- For angles, it means that they have the same measure: $\angle A \cong \angle B$ is the same as $m \angle A=m \angle B$.
- For triangles, it means that the corresponding sides are equal and corresponding angles are equal: $\triangle A B C \cong \triangle A^{\prime} B^{\prime} C^{\prime}$ is the same as
$A B=A^{\prime} B^{\prime}, B C=B^{\prime} C^{\prime}, A C=A^{\prime} C^{\prime}, m \angle A=m \angle A^{\prime}, m \angle B=m \angle B^{\prime}, m \angle C=m \angle C^{\prime}$.
Note that for triangles, the notation $\triangle A B C \cong \triangle A^{\prime} B^{\prime} C^{\prime}$ not only tells that these two triangles are congruent, but also shows which vertex of the first triangle corresponds to which vertex of the second one. For example, $\triangle A B C \cong \triangle P Q R$ is not the same as $\triangle A B C \cong$ $\triangle Q P R$.


## 2. Congruence tests for triangles

By definition, to check that two two triangles are congruent, we need to check that corresponding angles are equal and corresponding sides are equal; thus, we need to check 6 equalitites. However, it turns out that in fact, we can do with fewer checks.

Angle-Side-Angle Congruence Axiom (ASA) If $m \angle A=m \angle A^{\prime}, m \angle B=m \angle B^{\prime}$ and $A B=A^{\prime} B^{\prime}$, then $\triangle A B C \cong \triangle A^{\prime} B^{\prime} C^{\prime}$.

Congruence Axiom (SSS) If $A B=A^{\prime} B^{\prime}, B C=B^{\prime} C^{\prime}$ and $A C=A^{\prime} C^{\prime}$ then $\triangle A B C \cong$ $\triangle A^{\prime} B^{\prime} C^{\prime}$.

Congruence Axiom (SAS) If $A B=A^{\prime} B^{\prime}, A C=A^{\prime} C^{\prime}$ and $m \angle A=m \angle A^{\prime}$, then $\triangle A B C \cong \triangle A^{\prime} B^{\prime} C^{\prime}$.

## 3. Isosceles triangles

A triangle is isosceles if two of its sides have equal length. The two sides of equal length are called legs; the point where the two legs meet is called the apex of the triangle; the other two angles are called the base angles of the triangle; and the third side is called the base.

While an isosceles triangle is defined to be one with two sides of equal length, the next theorem tells us that is equivalent to having two angles of equal measure.

Theorem If $\triangle A B C$ is isosceles, with base $A C$, then $m \angle A=m \angle C$.
Conversely, if $\triangle A B C$ has $m \angle A=m \angle C$, then it is isosceles, with base $A C$.
In any triangle, there are three special lines from each vertex. In $\triangle A B C$, the altitude from $A$ is perpendicular to $B C$ (it exists and is unique by Theorem 7); the median from $A$ bisects $B C$ (that is, it crosses $B C$ at a point $D$ which is the midpoint of $B C$ ); and the angle bisector bisects $\angle A$ (that is, if $E$ is the point where the angle bisector meets $B C$, then $m \angle B A E=m \angle E A C)$.

For general triangle, all three lines are different. However, it turns out that in an isosceles triangle, they coincide.
Theorem If $B$ is the apex of the isosceles triangle $A B C$, and $B M$ is the median, then $B M$ is also the altitude, and is also the angle bisector, from $B$.

## Homework

1. Do problems 4,5 on page 143 , problems $6,7,8$ on page 145 , problems 16,17 , 18 on page 146 in the textbook.
2. Let $\triangle A B C$ be such that all sides have equal length. Prove that then $m \angle A=m \angle B=$ $m \angle C=60^{\circ}$. [Such a triangle is called equilateral.]
3. Prove that if $\triangle A B C$ has $m \angle A=m \angle C$, then it is isosceles, with base $A C$.
4. Let $A B C D$ be a quadrilateral such that $A B=$ $B C=C D=A D$ (such a quadilateral is called rhombus). Let $M$ be the intersection point of $A C$ and $B D$.
(a) Show that $\triangle A B C \cong \triangle A D C$
(b) Show that $\triangle A M B \cong \triangle A M D$
(c) Show that the diagonals are perpendicular and that the point $M$ is the midpoint of each of the diagonals.

5. The following method explains how one can find the midpoint of a segment $A B$ using a ruler and compass:

- Choose radius $r$ (it should be large enough) and draw circles of radius $r$ with centers at $A$ and $B$.
- Denote the intersection points of these circles by $P$ and $Q$. Draw a line $\overleftrightarrow{P Q}$.
- Let $M$ be the intersection point of $\overleftrightarrow{P Q}$ and $\overleftrightarrow{A B}$. Then $M$ is the midpoint of $A B$.


Can you justify this method, i.e., prove that so constructed point will indeed be the midpoint of $A B$ ? You can use the defining property of the circle: for a circle of radius $r$, the distance from any point on this circle to the center is exactly $r$.[Hint: use the previous problem]
6. The following method explains how one can construct a perpendicular from a point $P$ to line $l$ using a ruler and compass:

- Choose radius $r$ (it should be large enough) and draw circle of radius $r$ with center at $P$.
- Let $A, B$ be the intersection points of this circle with $l$. Find the midpoint $M$ of $A B$ (using the method of the previous problem). Then $\overleftrightarrow{M P} \perp l$.
Can you justify this method, i.e., prove that so constructed $\overleftrightarrow{M P}$ will indeed be perpendicular to $l$ ?

7. In the triangle $\triangle A B C$, let $M$ be a point on side $A B$. Prove that $m \angle B M C>m \angle A$.
8. What is the angle between the hour hand and minute hand of a clock at 11:10 AM? (Mathcounts)
9. Line $L$ is the perpendicular bisector of $P Q$. Find $a, b$ and $c$.

10. Line $K$ is the perpendicular bisector of $P Q$, line $L$ is the perpendicular bisector of $Q M$. Show that $O P=O M$.

