Math 5: Handout 22
Geometry 3.

Congruence tests for triangles

Recall that by definition, to check that two two triangles are congruent, we need to check that corresponding angles are equal and corresponding sides are equal; thus, we need to check 6 equalities. However, it turns out that in fact, we can do with fewer checks.

Axiom 1 (SSS Rule)**.** *If $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$ then $\triangle ABC \cong \triangle A'B'C'$.*

We can also try other ways to define a triangle by three pieces of information, such as a side and 2 angles (ASA), three angles (AAA), or two sides and an angle. For the two sides and an angle, there are two versions, one in which the two sides are adjacent to the given angle (SAS) and the other in which one of the given sides is opposite to the given angle (SSA). It turns out that ASA and SAS do indeed define a triangle:

Axiom 2 (Angle-Side-Angle Rule)**.** *If $\angle A = \angle A'$, $\angle B = \angle B'$ and $AB = A'B'$, then $\triangle ABC \cong \triangle A'B'C'$.*

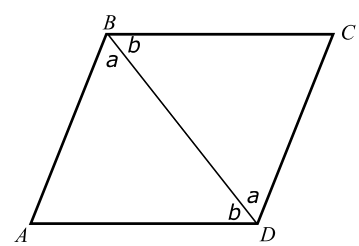
This rule is commonly referred to as ASA rule.

Axiom 3 (SAS Rule)**.** *If $AB = A'B'$, $AC = A'C'$ and $\angle A = \angle A'$, then $\triangle ABC \cong \triangle A'B'C'$.*

These rules — and congruent triangles in general — are very useful for proving various properties of geometric figures. As an illustration, we prove the following useful result.

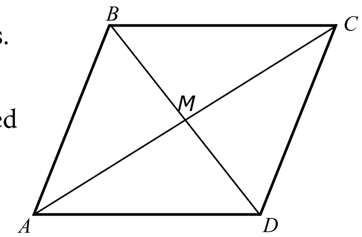
Theorem. *Let $ABCD$ be a parallelogram. Then $AB = CD$, $BC = AD$, i.e. the opposite sides are equal.*

Proof. Let us draw diagonal BD. Then the two angles labeled by letter a in the figure are equal as alternate interior angles (because $AB \parallel DC$); also, two angles labeled by letter b are also equal. Thus, triangles $\triangle ABD$ and $\triangle CDB$ have a common side BD and the two angles adjacent to it are the same. Thus, by ASA, these two triangles are congruent, so $AD = BC$, $AB = CD$. $\square$



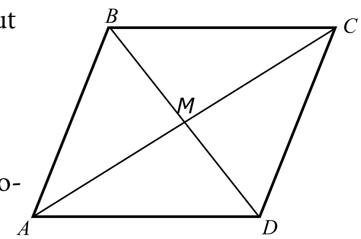
Homework

1. Solve the equation $3x + 3 = \frac{1}{2}x + 13$
2. a. Explain why in a rectangle, opposite sides are equal.
   b. Show that a diagonal of a rectangle cuts it into two congruent triangles.
3. Let $ABCD$ be a parallelogram, and let M be the intersection point of the diagonals.



a. Show that triangles $\triangle AMB$ and $\triangle CMD$ are congruent. [Hint: use the theorem proved in class, that the opposite sides are equal, and ASA.]
b. Show that $AM = CM$, i.e., M is the midpoint of diagonal AC.

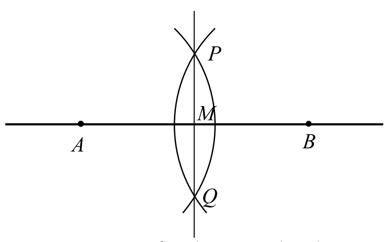
4. Let $ABCD$ be a quadrilateral such that sides AB and CD are parallel and equal (but we do not know whether sides AC and BD are parallel).



a. Show that triangles $\triangle AMB$ and $\triangle CMD$ are congruent.
b. Show that sides AC and BD are indeed parallel and therefore $ABCD$ is a parallelogram.

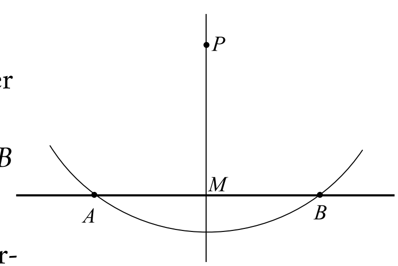
5. The following method explains how one can find the midpoint of a segment AB using a ruler and compass:

- Choose radius r (it should be large enough) and draw circles of radius r with centers at A and B.
- Denote the intersection points of these circles by P and Q. Draw the line PQ.
- Let M be the intersection point of lines PQ and AB. Then M is the midpoint of AB.



Can you justify this method, i.e., prove that so constructed point will indeed be the midpoint of AB? You can use the defining property of the circle: for a circle of radius r, the distance from any point on this circle to the center is exactly r.[Hint: $APBQ$ is a rhombus, so we can use problem 4 from the previous HW.]

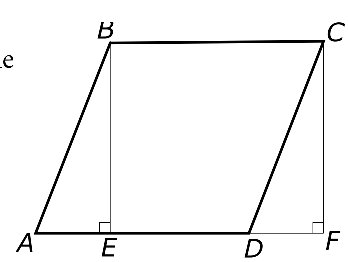
6. The following method explains how one can construct a perpendicular from a point P to line l using a ruler and compass:



- Choose radius r (it should be large enough) and draw circle of radius r with center at P.
- Let A, B be the intersection points of this circle with l. Find the midpoint M of AB (using the method of the previous problem). Then MP is perpendicular to l.

Can you justify this method, i.e., explain why so constructed MP will indeed be perpendicular to l?

7. Let $ABCD$ be a parallelogram, and let BE, CF be perpendiculars from B, C to the line AD.



a. Show that triangles $\triangle ABE$ and $\triangle DCF$ are congruent.
b. Show that the area of parallelogram is equal to height×base, i.e. $BE \times AD$.