Math 5: Handout 19
Factorials and Permutations. Monty-Hall Problem.

Factorials and permutations

Recall from last time: if we are choosing k objects from a collection of n so that a)order matters and b)no repetitions allowed, then there are

$$_kP_n = n(n-1)\dots \quad (k \text{ factors})$$

ways to do it.

In particular, if we take $k = n$, it means that we are selecting one by one all n objects — so this gives the number of possible ways to put n objects in some order:

$$n! = {}_nP_n = n(n-1)\cdot \dots \cdot 2 \cdot 1$$

(reads n factorial).

For example: there are 52! ways to mix the cards in the usual card deck.

Note that the number $n!$ grow very fast: $2! = 2$, $3! = 6$, $4! = 2 \cdot 3 \cdot 4 = 24$, $5! = 120$, $6! = 620$

Using factorials, we can give a simpler formula for $_kP_n$:

$$_kP_n = \frac{n!}{(n-k)!}$$

For example:

$$_4P_6 = 6 \cdot 5 \cdot 4 \cdot 3 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = \frac{6!}{2!}$$

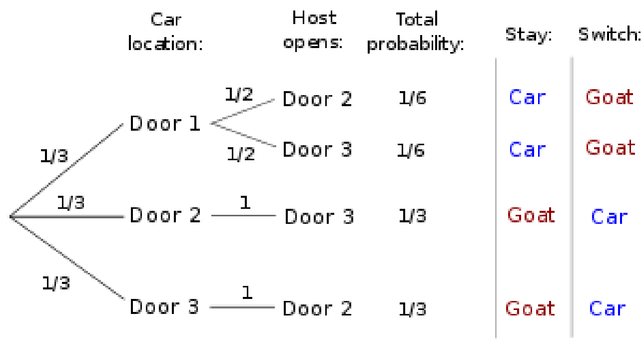
Monty-Hall Problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

A car and two goats are arranged behind three doors, with the position of the car being random and uniformly distributed among the three doors, and then the player initially picks a door. Assuming the player's initial pick is Door 1 (the same analysis applies for any other door the player picks), then according to the problem statement in which the host must always open a door after the player chooses one, then there are three equally likely cases:

- The car is behind Door 3 and the host must open Door 2: Probability = 1/3
- The player originally picked the door hiding the car. The game host must open one of the two remaining doors randomly, so there are two subcases
  - The car is behind Door 1 and the host opens Door 2: Probability = 1/6
  - The car is behind Door 1 and the host opens Door 3: Probability = 1/6
- The car is behind Door 2 and the host must open Door 3: Probability = 1/3

These cases and the total probability of each of them occurring are shown in the figure below. If the host has opened Door 3 switching wins in the 1/3 case where the car is behind Door 2 and loses in one 1/6 subcase where the car is behind Door 1, hence **switching wins with probability 2/3**.



Homework

1. A group of 6 club members always dine at the same table in the club; there are exactly 6 chairs at the table. They decided that each day, they want to seat in a different order. Can they keep this for a year? Two years?
2. How many ways are there to seat 15 students in a classroom which has 15 chairs? If the room has 25 chairs?
3. A small theater has 50 seats. One day, all 50 seats were taken – but the people took seats at random, paying no attention to what was written on their ticket.

    a. What is the probability that everyone was sitting in the right seat (i.e., the one written in his ticket)?

    *b. What is the probability that no person was sitting in the right seat?

4. A puzzle consists of 9 small square pieces which must be put together to form a 3×3 square so the pattern matches (this kind of puzzles is actually quite hard to solve!). It is known that there is only one correct solution. If you started trying all possible combinations at random, doing one new combination a second, how long will it take you to try them all?
5. a. How many 5s are there in the prime factorization of the number 100!? How many 2s?

    b. In how many zeroes does the number 100! end?
6. 10 people must form a circle for some dance. In how many ways can they do this?

*7. Here is another question similar to the Monty Hall question discussed today. You know that the family next door has two children. You met one of them, and he is a boy. What is the probability that the other one is a boy, too?