Math 5: Handout 18
Beginning Probability – 2.

Product rule

Question: we roll two dice. What is the probability of rolling a 5 and a 6?

Answer: There are two ways of getting a 5 and a 6: as pair (5, 6) (5 on die number 1, 6 on die number 2) or as (6, 5) (6 on die number 1, 5 on die number 2). Thus, the answer is $\frac{2}{36}$.

Question: we roll two dice. What is the probability of getting sum of two numbers equal to 4?

Answer: there 3 ways of getting sum 4: (1, 3), (2, 2), (3, 1). Thus the probability is $\frac{3}{36} = \frac{1}{12}$.

Question. If toss a coin 10 times, what is the probability that all will be heads?

Answer. $\left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}}$ (using calculator, one can compute that it is $1/1024 \approx 0.001$, or $1/10$ of 1%).

Question. If toss a coin 10 times, what is the probability that all will be tails?

Answer. The same.

Question. If we toss a coin 10 times, what is the probability that **at least one** will be heads?

Answer. Unfortunately, there are very many combinations which give at least one heads. In fact, it is easier to say which combinations **do not** give at least one heads: there is exactly one such combination, all tails; probability of getting this combination is, as we computed, $1/2^{10} = \frac{1}{1024}$. The remaining combinations will give at least one heads; thus probability of getting at least one heads is $1 - \frac{1}{1024} = \frac{1023}{1024} \approx 0.999$.

Percentages and fractions

So far we have mostly expressed probabilities as fractions. They can also be written as decimal numbers (between 0 and 1): for example, $\frac{1}{5} = \frac{2}{10} = 0.2$. It is also common to express probabilities as percentages: by definition,

$$1\% = \frac{1}{100} = 0.01$$

so $x\% = \frac{x}{100}$. For example, $3\% = \frac{3}{100} = 0.03$, and $1.5\% = \frac{1.5}{100} = \frac{15}{1000} = 0.015$.

This conversion is necessary when you multiply probabilities as the following example shows:

Question. The probability of winning in a certain game is $p = 5\%$. What is the probability of winning two times in a row?

Answer. According to multiplication rule it is $p \times p = p^2$. However, the answer $5\% \times 5\% = 25\%$ is wrong. Correct answer is $\frac{5}{100} \times \frac{5}{100} = \frac{25}{10{,}000} = 0.0025$.

To convert from decimals to percent, multiply by 100:

$$p = (p \times 100)\%$$

For example, $\frac{1}{5} = 0.2 = (0.2 \times 100)\% = 20\%$

Choosing with repetitions

Problem: how many 3-letter combinations can be formed using 26 letters of Latin alphabet?

Answer: there are 26 possibilities for the first letter, 26 for the second, and so on — so according to the product rule, there are $(26)^3$ possible combinations.

The same method of counting can be applied in more general situation: suppose we need to choose k items from a collection of n so that

- Order matters: choosing A, then B is different from choosing B, then A.
- Repetitions are allowed: same item can be used more than once (e.g., same letter may appear several times in a combination).

Then there are n^k ways to do it.

Choosing without repetitions

Problem: how many 3-letter combinations can be formed using 26 letters of Latin alphabet if no letter can be used more than once?

Answer: there are 26 possibilities for the first letter; after we have chosen the first letter, it leaves only 25 possibilities for the second letter; after choosing the second, we only have 24 possibilities left for the third. So the answer is $26 \times 25 \times 24$

The same method of counting can be applied in more general situation: suppose we need to choose k items from a collection of n so that

- Order matters: choosing A, then B is different from choosing B, then A.
- Repetitions are not allowed: no item can be used more than once.

Then there are $n(n-1)\dots(n-k+1)$ ways of doing it (the product has k factors). This number is usually denoted

$$ {}_kP_n = n(n-1)\dots(n-k+1) $$

Homework

1. If we roll two dice, what is the probability that the product of two numbers is a multiple of 3?
2. Recall that a roulette has 37 slots: 0 through 36. Among slots 1–36, half are red, the other half black (zero has no color). What is the probability of
   a. getting a red (on a single run of roulette)
   b. getting a red, then black, then 0 (on 3 successive runs)
   c. getting red 15 times in a row?
   d. getting this sequence of colors: RRRBRBRBBRBBRBR (also of length 15)?
3. A hunter is shooting ducks. Probability of hitting a duck with one shot is $p = 1/3$.
   a. What is the probability of missing the duck (with one shot)?
   b. He makes 5 shots. What is the probability that he misses all five?
   c. What is the probability that out of 5 shots, he will hit at least once? Will this probability double if he makes 10 shots? (You can use the calculator for computing the answers)
   d. What is the probability that out of 5 shots, he will hit exactly once? Will this probability double is he makes 10 shots?

   *(e.) What is the probability that out of 5 shots, he will hit at least twice? Will this probability double if he makes 10 shots? (You can use the calculator for computing the answers)

   *(f.) What is the probability that out of 5 shots, he will hit exactly twice? Will this probability double if he makes 10 shots? (You can use the calculator for computing the answers)
4. Supposing that there are equal chances of a boy or a girl being born, what is the probability that at least one of the first five babies born next Saturday morning at the St. Charles Hospital will be a girl? That all five will be girls?
5. At a fair, they offer you to play the following game: you are tossing small balls in a large crate full of empty bottles; if at least one of the balls lands inside a bottle, you win. Unfortunately, it is really impossible to aim, so the game is just a matter of luck (or probability theory): every ball you toss has a 20% probability of landing inside the bottle.
   a. If you are given three balls, what is the probability that all three will be hits? That all three will be misses? That at least one will be a hit?
   b. Same questions for five balls.
   c. They charge you 2 dollars for 3 balls, or 3 dollars for 5 balls. Which is a better deal? [Considering only from the point of view of the chances of winning, not the fun you are getting]
6. A license plate consists of 3 letters, followed by three digits. How many possible license plates are there?
7. In one kind of lottery, they put balls with numbers 1 through 100 in a bag and then draw six balls at random (drawn ball is put aside and not returned to the bag). To win the lottery, one needs to guess all six numbers in correct order. What is the probability of this?