TWO STRATEGY PROBLEMS

APRIL 14, 2019

1. Warm-up problem
  (a) How many permutations of a set of 100 elements there are which are a single cycle of length 100?
  (b) How many permutations of a set of 100 elements are there that contain a cycle of length 51?
  (c) How many permutations of a set of 100 elements are there that contain a cycle of length more than 50?

2. (This is a famous problem, suggested in 2003 by a Danish computer scientist Peter Bro Miltersen. It is a hard problem, but the previous problem gives a hint.)

The prison warden offers 100 death row prisoners, who are numbered from 1 to 100, a last chance. A room contains a cupboard with 100 drawers. The warden randomly puts one prisoner's number in each closed drawer. The prisoners enter the room, one after another. Each prisoner may open and look into 50 drawers in any order. The drawers are closed again afterwards. If, during this search, every prisoner finds his number in one of the drawers, all prisoners are pardoned. If just one prisoner does not find his number, all prisoners die. Before the first prisoner enters the room, the prisoners may discuss strategy — but may not communicate once the first prisoner enters to look in the drawers. The prisoners are not allowed to move numbers from one drawer to another or make any other changes.

What is the prisoners' best strategy?

Note: there is no strategy that guarantees the prisoners win, but there are strategies that offer a chance of survival significantly better than $(1/2)^{100}$.

3. (This problem is more recent, and I do not know its author. I found it in one of many internet communities dedicated to mathematical games and puzzles).

Alica and Bob are playing the following game. Each of them, in his own room, tosses a fair coin N times (where N is very large) and records the results. Then Alice select a number a between 1 and N and sends it to Bob; similarly, Bob selects a number b between 1 and N and send it to Alice. (This is done at the same moment — neither of them nows what number the other had chosen before making his own choice.)

After that they compare the results: if result of Bob's a-th toss was the same as result of Alice's b-th toss, they win.

Example:

Alice: TTHHHTHTT...

Bob: HTTHTTTHH...

Alice chooses number $a = 8$; Bob's 8th toss was heads

Bob chooses number $B = 4$; Alice's 4th toss was also heads. Thus, they win.

Now, the question: is there a strategy that is expected to give them chance of winning better than 50%?