PERMUTATIONS

APRIL 7, 2019

A **permutation** of some set S is a way of reordering, or permuting, elements of S. Mathematically, it can be described as a function $f\colon S \to S$ which is a bijection (one-to-one and onto, or invertible). We will only be discussing permutations of finite sets, usually the set $S = \{1, \dots, n\}$. In this case one can also think of a permutation as a way of permuting n items placed in boxes labeled $1, \dots, n$: namely, move item from box 1 to box $f(1)$, item from box 2 to $f(2)$, etc. The set of all permutations of $\{1, \dots, n\}$ is denoted by S_n.

Permutations can be composed in the usual way: $f \circ g(x) = f(g(x))$.

Notation: the permutation f which sends 1 to a_1, 2 to a_2, etc, is usually written as

$$\begin{pmatrix} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

An alternative way of writing permutations is using cycles. A **cycle** $(a_1 a_2 \dots a_k)$ is a permutation which sends a_1 to a_2, a_2 to a_3, …, a_n to a_1 (and leaves all other elements unchanged). For example, (123) is the permutation such that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$ and $f(a) = a$ for all other a. The same cycle can also be written as (231).

We can also consider products (i.e. compositions) of several cycles. For example, (123)(45) is a permutation such that $f(1) = 2$, $f(2) = 3$, $f(3) = 1$, $f(4) = 5$, $f(5) = 4$. It is also customary not to write cycles of length one: instead of writing (123)(4), we write just (123).

1. Fifteen students are meeting in a classroom which has 15 chairs numbered 1 through 15. The teacher requires that every minute they change seats following this rule:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3	5	10	8	11	14	15	6	13	1	4	9	7	2	12

(e.g., the student who was sitting in the chair number 1 would move to chair number 3). In how many minutes will the students return to their original seats?

2. Let $s \in S_9$ be a permutation of set $\{1, 2, \dots 9\}$. Can you find such an s so that $s^7 = 1$, i.e., repeating this permutation 7 times we get back the orginal order of elements? what about $s^{12} = 1$? $s^{11} = 1$?

3. n chairs are placed in a row; in each chair, there is a student. Two students who are sitting next to each other are allowed to switch seats. How many such operations are required to put students in reverse order?

4. n chairs are placed in a circle; in each chair, there is a student. Two students who are sitting next to each other are allowed to switch seats. How many such operations are required to put students in reverse order? [Reverse order here means that any two students who were sitting next to each other will still be sitting next to each other, but in opposite order.]

5. Each of n players in some board game has a playing card (all cards are different). They trade them: any two players can exchange their cards; however, at every game turn, every player is only allowed to trade the cards once.

Show that any possible exchange of cards, no matter how complicated, can be done in two turns (if the players all cooperate).

6. A deck of 52 cards is shuffled using the following operation:

a contiguous block of cards is removed from the deck (can be from the top, bottom, or from the middle of the deck) and inserted in some other place in the deck. The order of cards inside the block is not changed.

(a) Show that it is possible to completely reverse the order of the cards in the deck using 27 such operations

(b) Show that 17 operations are not enough

*(c) Show that 26 such operations are not enough.

7. n cards are placed on the table. You are allowed to switch places of two cards at a time. Is it possible to return to the original order after doing 4 such operations? after 5? after 2019?