DIOPHANTINE EQUATIONS

FEB 24, 2019

Equations where all variables are only alowed to take integer values are called *Diophintine equations*. There is no general method of solving them, but there are some useful tricks.

Simple diophantine equations

1. Solve the following equation: $2^x + 7 = y^2$

2. Find all positive integers x, y, z for which

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

3. Find all positive integers $x, y, z > 1$ for which

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > 1$$

Pythagorean triples

A triple of positive integers a, b, c such that

$$a^2 + b^2 = c^2$$

is called a *Pythagorean triple.* A simplest such triple is $3, 4, 5$.

A Pythagorean triple is called *primitive* if $\gcd(a, b, c) = 1$.

4. Let (a, b, c) be a primitve Pythagorean triple. Show that then one of the numbers a, b is even and the other is odd.

5. Let (a, b, c) be a primitve Pythagorean triple, with a even and b odd. Show that then one can find two relatively prime integers m, n such that $a = 2mn$, $b = m^2 - n^2$, $c = m^2 + n^2$. Use these formulas to construct some examples of Pythagorean triples.

Pell's equation

Pell's equation is the following Diophantine equation:

$$x^2 - dy^2 = 1,$$

where d is some given positive squarefree integer (i.e., not divisible by any square except 1).

Study of solutions of Pell's equation is closely related to the study of the following set

$$R = \{a + b\sqrt{d}, a, b \in \mathbb{Z}\} \subset \mathbb{R}$$

so we begin with some problems about R.

7. Show that R is closed under multiplication, addition.

8. For a number $r = a + b\sqrt{d} \in R$, denote $\overline{r} = a - b\sqrt{d}$ and $N(r) = r\overline{r} = a^2 - db^2$. Prove that $\overline{r_1 r_2} = \overline{r_1} \cdot \overline{r_2}$, and $N(r_1 r_2) = N(r_1)N(r_2)$.

9. Show that if $N(r) = \pm 1$, then r^{-1} is also in R.

10. Deduce from the previous problems that the set $U = \{r \in R \mid N(r) = 1\}$ is closed under multiplication and operation of taking inverses. In particular, if $r \in U$, then for anu integer n, we have $r^n \in U$.

11. Show that if a Pell's equation has at least one solution, then it has infinitely many solutions.

12. Construct infinitely many solutions of the following equations: $x^2 - 2y^2 = 1$; $x^2 - 5y^2 = 1$.

(In fact, it is known that for any d, Pell's equation has infinitely many solutions. Moreover, all of them can be obtained from a single solution (a, b) by writing $r = a + b\sqrt{d}$ and then considering $\pm r^n$ for $n \in \mathbb{Z}$. This, however, is much more difficult to prove.]