MORE NUMBER THEORY PROBLEMS

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Useful things to know from number theory

Here is a short listing of elementary results which can be useful when solving number theory problems. Detailed exposition can be found in many places, including SchoolNova class archive.

- Euclid algorithm. GCD and LCM. Fact: for given a, b, an integer m can be written in the form
$$m = ax + by$$
if and only if m is the multiple of $\gcd(a, b)$.
- Divisibility. If ax is divisible by b, and a, b are relatively prime, then x is divisible by b. If a, b are relatively prime, then any common multiple of a, b is a multiple of ab. In particular, $\operatorname{lcm}(a, b) = ab$.
- Prime factorization. Counting the number of divisors and sum of divisors from prime factorization.
- Congruences and remainders mod n. Invertible remainders. Remainders of $a^m \mod n$ repeat periodically.
- Chinese remainder theorem

Some problems from last time

1. What is the smallest positive integer n such that 2013^n ends in 01 (i.e. the rightmost two digits of 2013^n are 01)? What if we want the last three digits to be 001?

2. What is the largest positive integer that cannot be expressed as a sum of non-negative integer multiples of 13 and 17?

What is the largest positive integer that cannot be expressed as a sum of non-negative integer multiples of 13, 17 and 23?

More number theory problems

In all the problems below, all numbers are positive integers.

1. Find all solutions of $x^2 = y^2 + 23$

2. Find all solutions of $x^2 + 84x + 2008 = y^2$

3. Find all primes p such that $71p + 1$ is a perfect square.

4. For which n both $n + 76$ and $n - 76$ are perfect cubes?

5. Find the largest k such that 3^{11} can be written as sum of k consecutive integers.

6. Let $n = 1001$. How many numbers are divisors of n^3 but not divisors of n^2?

7. How many divisors does the number 17! have?

8. (a) Show that $2^{2019} + 1$ is divisible by 3.

(b) Suppose that $2^n + 1$ is a prime for some positive integer n. Show that n must be a power of 2.

9. How many pairs of numbers (a, b) satisfy
$$a^2 + b^2 = ab(a + b)$$

10. [From 2019 AMC12B] How many sequences of 0s and 1s of length 19 are there that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s?

***11.** Let $p_1 \dots P_n$ be distinct primes greater than 3. Prove that then the number
$$2^{p_1 \cdots p_n} + 1$$
has at least 4^n divisors.