SOME NUMBER THEORY PROBLEMS

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Useful things to know from number theory

Here is a short listing of elementary results which can be useful when solving number theory problems. Detailed exposition can be found in many places, including SchoolNova class archive.

- Euclid algorithm. GCD and LCM. Fact: for given a, b, an integer m can be written in the form
$$m = ax + by$$
if and only if m is the multiple of $\gcd(a, b)$.
- Divisibility. If ax is divisible by b, and a, b are relatively prime, then x is divisible by b. If a, b are relatively prime, then any common multiple of a, b is a multiple of ab. In particular, $\operatorname{lcm}(a, b) = ab$.
- Prime factorization. Counting the number of divisors and sum of divisors from prime factorization.
- Congruences and remainders mod n. Invertible remainders. Remainders of $a^m \mod n$ repeat periodically.
- Chinese remainder theorem

Simple problems

1. How many four-digit numbers N have the property that the three-digit number obtained by removing the leftmost digit is equal to one ninth of N?

2. Find the remainder upon the division of 17^{2019} by 7.

3. In how many zeroes does the number 100! end?

4. Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7

5. How many perfect squares are divisors of of the product $1! \cdot 2! \cdots 9!$?

6. If p, q and r are primes with $pqr = 7(p + q + r)$, find $p + q + r$.

7. (a) Show that remainder upon division of $2^a - 1$ by $2^b - 1$ is equal to $2^r - 1$, where r is the remainder upon division of a by b. [Hint: if $a = qb + r$, then $2^a - 1 = 2^{qb+r} - 2^r + 2^r - 1$.]
(b) Use Euclid's algorithm to show that $\gcd(2^m - 1, 2^n - 1) = 2^{\gcd(m,n)} - 1$
(c) Find $\gcd(2^{2019} - 1, 2^{1968} - 1)$

Harder problems

For those who know how to do all the previous ones.

1. What is the smallest positive integer n such that 2013^n ends in 01 (i.e. the rightmost two digits of 2013^n are 01)? What if we want the last three digits to be 001?

2. What is the largest positive integer that cannot be expressed as a sum of non-negative integer multiples of 13 and 17?

What is the largest positive integer that cannot be expressed as a sum of non-negative integer multiples of 13, 17 and 23?