PROBABILITIES 2

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Let x be a random variale, i.e. a quantity which can take different real values depending on some random event (such as tossing a coin). If the possible values are $x_1, x_2, \ldots$, appearing with probabilities $p_1, p_2, \ldots$, respectively, then the *expected value* of x is defined to be

$$Ex = \sum x_i p_i.$$

1. A newly bought lightbulb will be good with probability of 95%.

How many lightbulbs you need to buy to have 5 good bulbs with probability at least 99%?

2. One tosses a coin 10 times. What is the probability that there will be no two heads in a row?

[Hint: let $f(n)$ be the number of combinations of n coin tosses which have no two heads in a row. Compute $f(n)$ for small values of n and try to guess the pattern.]

3. A movie theater has n seats. Today it is sold out: all seats are sold. Each of n tickets has a seat number on it. However, the first person who comes to the theater is careless, so instead of taking his assigned seat, he takes a random seat. After this, each next person coming to the theater tries to take hsi seat, but if it is already taken, they take a random seat.

What is the probability that the last person will be seated in his assigned seat?

4. n people are standing in the line, one after another. If you look at the line from the beginning, you will only see some of them: if a taller person stands before a shorter one, he blocks the view, so you will not be able to see the shorter person.

What is the expected number of people you will see?

[Hint: try to see how this expected number changes when you increase n by 1.]

5. A frog jumps between three pads, labeled A, B, and C. Every minute it jumps from the pad it currently occupies to one of the other pads, and the probabilities are as follows:

- From pad A: jumps to B with probability $1/2$, and to C also with probaiblity $1/2$
- From pad B: jumps to A with probability $1/3$, and to C with probaiblity $2/3$
- From pad C: jumps to A with probability $1/3$, and to B with probaiblity $2/3$

If you check on the frog at random moment, what is the probability of finding the frog on pad A? on pad B? on pad C?

[Hint: try to write some relations between these three probabilities.]

6. We have 100 letters, addressed to different people, and 100 envelopes carrying the addresses. A lazy clerk puts the letters in the envelopes at random.

What is the probability that at least one letter will reach its addressee? that no letter will reach its addressee?

[Hint: when doing this problem, you might need the so-called inclusion-exclusion principle: if A_1, A_2, ... are subsets of some set A, then

$$\begin{aligned} |A_1 \cup \cdots \cup A_n| = &|A_1| + \cdots + |A_n| \\ &- |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots \\ &+ |A_1 \cap A_2 \cap A_3| + \ldots \\ &\ldots \end{aligned}$$

The second line has all possible double intersections, the third line, all possible triple intersections, and so on.]