FORMAL POWER SERIES AND GENERATING FUNCTIONS

DECEMBER 2, 2018

Formal power series

A formal power series is an expression of the form

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

A special case of a formal power series is a polynomial.

Note that it is not a function: in generla, we can't substitute a numerical value for x, as it would give an infinite sum. However, we can still perfectly well add and multiply such formal series.

For example,

$$\begin{aligned}(1-x)(1+x+x^2+x^3+\dots) &= 1\\ (1+x+x^2+x^3+\dots)^2 &= 1+2x+3x^2+\dots\end{aligned}$$

Because of this, we will frequently write $\frac{1}{1-x}$ meaning the power series $1 + x + x^2 + \dots$

Generating function

Let a_n, $n = 0, 1, \dots$ be a sequence. The *generating function* for this sequence is the formal power series

$$F(x) = a_0 + a_1x + a_2x^2 + \dots$$

It turns out that in many cases, it is easier to find the generating function then the formula for the terms of the sequence — and of course, knwing the generating fucntion determines the sequence.

Problems

1. For every non-empty subset $S \subset \{1, \dots, n\}$, denote by $p(S)$ the product of inverses of elements of S. What is the sum of $p(S)$ over all such subsets S?

2. Find the product of the following formal power series:

$$(1+x+x^2+x^3+\dots)(1-x+x^2-x^3+\dots)$$

3. Compute the generating function of the following sequences:
(a) $a_n = {}_{m+n}C_m$ (where m is some fixed number).
(b) $a_n = {}_nC_m$

4. (This problem is for people familiar with the notion of derivative.)
Prove that for a formal power series $f(x) = \sum a_n x^n$, we have

$$a_n = \frac{f^{(n)}(0)}{n!}$$

5. Let a_n be the number of solutions of equation $x_1 + x_2 + \dots + x_k = n$ (all x_i must be non0-negative integers). Let $F(x)$ be the generating function for this sequence.
(a) Prove

$$F(x) = (1+x+x^2+\dots)^k = \frac{1}{(1-x)^k}.$$

(b) If you have done the previous problem, use it to find a formula for a_n.

6. Let $F(x) = F_0 + F_1x + F_2x^2 + \dots$ be the generating series for the Fibonacci numbers: $F_0 = 0, F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$. Prove that

$$F(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\Big(\frac{1}{1-\Phi x} - \frac{1}{1-\bar{\Phi} x}\Big)$$

where

$$\Phi = \frac{1+\sqrt{5}}{2}, \qquad \bar{\Phi} = \frac{1-\sqrt{5}}{2}$$

and use it to find an explicit formula for F_n.

[Hint: can you prove that $F(x) = (x + x^2)F(x)$?]

7. Let $d(n)$ be the number of ways to write n as a sum of **distinct** non-negative integers, and $l(n)$ – the number of ways to write n as a sum of **odd** non-negative integers. (In both cases, order of summands doesn't matter). By definition, we let $d(0) = l(0) = 1$.

Prove:

(a) $d(0) + d(1)x + d(2)x^2 + \cdots = (1 + x)(1 + x^2)(1 + x^3)\ldots$

(b) $l(0) + l(1)x + l(2)x^2 + \cdots = (1 - x)^{-1}(1x^3)^{-1}(1x^5)^{-1}\ldots$

(c) $l(n) = d(n)$.