PIGEONHOLE PRINCIPLE

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The pigeonhole principle

If you put n items in m boxes, with $n > m$, then at least one box will have more than one item.

Generalization
If $n > km$ objects are put in m boxes, then at least one box will have more than k objects.

Problems

1. If we have a group of $n > 1$ people, some of whom shake hands with each other, than there is among them a pair of people who have shaken hands with the same number of people.

2. Any subset of size 6 from the set $S = \{1, 2, \dots, 9\}$ must contain two elements whose sum is 10.

3. Let A be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, . . . , 100. Prove that there must be two distinct integers in A whose sum is 104. [Actually, 20 can be replaced by 19.]

4. Five points are placed inside an equilateral triangle whose side has length one unit. Show that two of them may be chosen which are less than one half unit apart. What if the equilateral triangle is replaced by a square whose side has length of one unit?

5. Given 5 points in the plane with integer coordinates, prove that there is a pair of points such that the midpoint of the segment connecting them also has integer coordinates.

6. Consider the sequence of numbers 1, 11, 111, 1111, . . . ,
(a) Prove that among these numbers there are two whose difference is divisible by 179
*(b) Prove that one of these numbers is divisible by 179.

7. Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint non-empty subsets whose members have the same sum.
[This problem is from 1972 IMO, but it is one of the simpler IMO problems. As a hint, try first finding two different such subsets without requiring that they be disjoint.]

8. Given any $n + 1$ integers between 1 and $2n$, show that one of them is divisible by another. Is this best possible, i.e., is the conclusion still true for n integers between 1 and $2n$?

9. (This is not related to pigeonhole principle :)
Prove that in the figure below, centers of three circles O_1, O_2, O_3 lie on the same line.