ADVANCED PROBLEM SOLVING CLUB
SELECTED HMMT PROBLEMS

OCTOBER 21, 2018

All problems are selected from November HMMT contests, which consist of four sections: General (Individual), Theme, Team, and Guts.

General

1. Plot points A, B, C at coordinates $(0,0)$, $(0,1)$, and $(1,1)$ in the plane, respectively. Let S denote the union of the two line segments AB and BC. Let X_1 be the area swept out when Bobby rotates S counterclockwise 45 degrees about point A. Let X_2 be the area swept out when Calvin rotates S clockwise 45 degrees about point A. Find $X_1 + X_2$. (2013, problem 2)

2. Consider all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(f(x) + 2x + 20) = 15$. Call an integer n *good* if $f(n)$ can take any integer value. In other words, if we fix n, for any integer m, there exists a function f such that $f(n) = m$. Find the sum of all good integers x. (2015, problem 6)

3. If a and b satisfy the equations $a + \frac{1}{b} = 4$ and $\frac{1}{a} + b = \frac{16}{15}$, determine the product of all possible values of ab. (2016, problem 1)

4. Five equally skilled tennis players named A, B, C, D, and E play in a round robin tournament, such that each pair of people play exactly once, and there are no ties. In each of the ten games, the two players both have a 50% chance of winning, and the results of the games are independent. Compute the probability that there exist four distinct players P_1, P_2, P_3, P_4 such that P_i beats P_{i+1} for $i = 1, 2, 3, 4$. (We denote $P_5 = P_1$). (2017, problem 10)

Theme

1. Consider a 4×4 grid of squares. Aziraphale and Crowley play a game on this grid, alternating turns, with Aziraphale going first. On Aziraphales turn, they may color any uncolored square red, and on Crowleys turn, they may color any uncolored square blue. The game ends when all the squares are colored, and Aziraphales score is the area of the largest closed region that is entirely red. If Aziraphale wishes to maximize their score, Crowley wishes to minimize it, and both players play optimally, what will Aziraphales's score be? (2015, problem 4)

2. Consider a 7×7 grid of squares. Let $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{1, 2, 3, 4, 5, 6, 7\}$ be a function; in other words, $f(1), f(2), ..., f(7)$ are each (not necessarily distinct) integers from 1 to 7. In the top row of the grid, the numbers from 1 to 7 are written in order; in every other square, $f(x)$ is written where x is the number above the square. How many functions have the property that the bottom row is identical to the top row, and no other row is identical to the top row? (2015, problem 7)

3. Consider a 9×9 grid of squares. Haruki fills each square in this grid with an integer between 1 and 9, inclusive. The grid is called a super-sudoku if each of the following three conditions hold:
• Each column in the grid contains each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once.
• Each row in the grid contains each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once.
• Each 3×3 subsquare in the grid contains each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once.
How many possible super-sudoku grids are there? (2015, problem 9)

Guts

1. Three ants begin on three different vertices of a tetrahedron. Every second, they choose one of the three edges connecting to the vertex they are on with equal probability and travel to the other vertex on that edge. They all stop when any two ants reach the same vertex at the same time. What is the probability that all three ants are at the same vertex when they stop? (2015, problem 24)

2. Let $ABCD$ be an isosceles trapezoid with parallel bases $AB = 1$ and $CD = 2$ and height 1. Find the area of the region containing all points inside $ABCD$ whose projections onto the four sides of the trapezoid lie on the segments formed by AB, BC, CD and DA. (2016, problem 15)

3. Find the remainder when $1^2 + 3^2 + 5^2 + \ + 99^2$ is divided by 1000. (2013, problem 9)

Team

1. Let π be a permutation of the numbers from 2 through 2012 (i.e., π is a function such that, for example, $\pi(5) = 127$, $\pi(6) = 331$, etc). Find the largest possible value of $log_2(2) \cdot log_3(3) \cdot ... \cdot log_{2012}(2012)$. (2012, problem 4)

2. Consider five-dimensional Cartesian space $\mathbb{R}^5 = \{(x_1, x_2, x_3, x_4, x_5) | x_i \in \mathbb{R}\}$, and consider the hyperplanes with the following equations:
- $x_i = x_j$ for every $1 \leq i < j \leq 5$;
- $x_1 + x_2 + x_3 + x_4 + x_5 = 1$;
- $x_1 + x_2 + x_3 + x_4 + x_5 = 0$;
- $x_1 + x_2 + x_3 + x_4 + x_5 = 1$.

Into how many regions do these hyperplanes divide $\mathbb{R}^5$? (2017, problem 6)