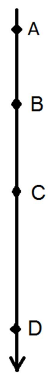
Conservative forces.

We spoke about this last year, but not everyone was there with us. And it is never bad idea to recollect an old story in science and repeat it to yourself – maybe you see something new in it now….

Let's take another look at the work done by the force of gravity $m\vec{g}$. This force looks vertically down, as you know, so let us consider this segment on the left and write down some formulas for work of this force between particular points.



Suppose the object moves from point A to point B on the picture, this is the work of the gravity: $mg \times AB = mg\ (x_B - x_A)$.

If it moves from B to C, this is the work: $mg \times BC = mg\ (x_C - x_B)$. .

And from C to D: $mg \times CD = mg\ (x_D - x_C)$. .

Now, if returns back from D to A, the work is $mg \times DA = mg\ (x_A - x_D)$ (of course this is a sum of works from D to C, C to B and B to A). Obviously this last expression is negative. So what is the total work performed by the gravity force on the body while the body moves from along this path ABCDA ? Let's add up:

$$mg\ (x_B - x_A) + mg\ (x_C - x_B) + mg\ (x_D - x_C) + mg\ (x_A - x_D) = 0$$

(I skipped opening parenthesis, I am sure you know how to do that)… It is zero! Actually, such work would be zero if the body traveled even a more sophisticated way, say ABCBCDBCA – as long as you end where you started, this work is zero! The forces which have this property (work along any closed trajectory – the one that ends where it started - is zero) are called ***conservative forces.***

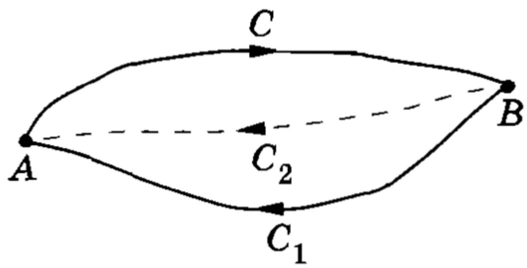
You also may remember a Homework problem where you proved equivalence of the above definition and another definition: *"conservative force" is a force for which work between points A and B does not depend on the trajectory, but only on the position of the initial point A and final point B.*

Now something new which we did not tell you before. We spoke about $m\vec{g}$. But same holds about the full Newton's law

$$\vec{F} = G\frac{mM}{R^2}\frac{\vec{R}}{R}$$

This force is conservative! Actually, any central force is conservative ! So, Coulomb force is conservative too. Conservative forces are sometimes also called "potential forces", two terms are synonyms.

Potentiality of Coulomb force.



Potentiality of Coulomb forces follows directly from the law of conservation of energy. Suppose for a second that on the picture on the left the work of Coulomb force done to move a charge along curve ACB is not equal to the inverse of the work along any other curve (BC_1A or BC_2A). Then looping the charged particle from A back to A you can produce energy from nothing – this means you have a Perpetuum Mobile.

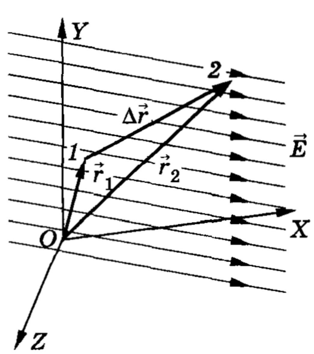
Potential Energy of Charged Particle.

Same logic as in mechanics of gravity allows us to replace speaking about the work of Coulomb force by speaking about the potential energy:

$$A = -(W_{p2} - W_{p1})$$

where W_{p2} is a potential energy in the end point of the trajectory, and W_{p1}- the potential energy in the starting point. Remember how energy is the ability to perform work? So, for example, energy decreasing means $\left(W_{p2} - W_{p1}\right) < 0$ and thus the work is positive – it was performed.

Potential Energy in Uniform field.

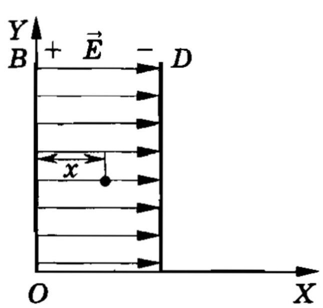


$$A = \vec{F} \cdot \Delta\vec{r} = q\vec{E} \cdot (\vec{r}_2 - \vec{r}_1) = q\vec{E} \cdot \vec{r}_2 - q\vec{E} \cdot \vec{r}_1$$

And comparing to the expression from the last paragraph:

$$W_p = -q\vec{E} \cdot \vec{r}$$

In particular, uniform field is produced between two infinite flat parallel planes



$$W_p = -q(E_x x + E_y y + E_z z) = -qE_x x$$

This formula is a twin sister of our *mgh* expression from last year.

Potential Energy of Interaction of Two Point Charges.

Think about a charge Q creating electric field around itself. Consider charge q which moves from distance r_1 from Q to distance r_2 from it. In general it is hard to compute the work of Coulomb force in this case – the force changes as the distance changes. If you split the trajectory onto small pieces then the work along the trajectory is the sum of works on these small pieces, but to compute this sum you need to know how to compute integrals.

In the same time, if the distance to which the q moved is much smaller than the distance between charges (i.e. $|r_2 - r_1| \ll r_1$) then we can neglect the changing of the force, assuming that the radius is a geometric mean of initial and final radii and the force is thus in Gaussian units

$$F = \frac{qQ}{r_1 r_2}$$

and its work is

$$A = \frac{qQ}{r_1 r_2}(r_2 - r_1) = \frac{qQ}{r_1} - \frac{qQ}{r_2}$$

(obviously only radial motion matters for work computation – scalar product of force and displacement is zero when you move on the sphere with a fixed radius). You see that the choice of geometric mean was dictated by our desire to have something of the form

$$A = -(W_{p2} - W_{p1})$$

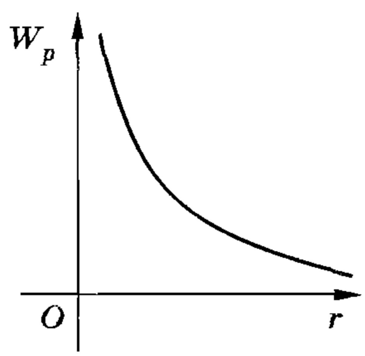
Thus the potential energy is

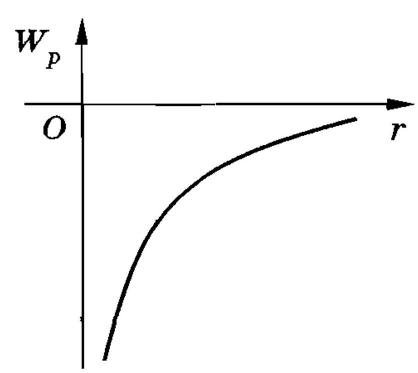
$$W_p = \frac{qQ}{r}$$

of course this is not unique choice.

$$W_p = \frac{qQ}{r} + C$$

with any constant C would also work. But we prefer to have zero energy at infinity, as on these graphs (for same sign and opposite sign charges):





Now suppose you really study the field, created by Q. You see how the potential energy of q in this field, divided by q itself, is a property of the field, not of the test charge q

$$\phi = \frac{W_p}{q} = \frac{Q}{r}$$

Same holds for uniform field:

$$\phi = \frac{W_p}{q} = -E_x x$$

Again – there are no memories of q in this expression….

This characteristic of the field is called the ***potential***.

Homework problems.

(1) *Three balls with charge q and mass m each are in the vertices of equilateral triangle with side equal to a. When you release them and let them fly apart, what would be the maximal speeds which they will achieve?*

(2) *Positive charge Q is uniformly distributed over the ring of radius R. In the center of the ring we place a negatively charged particle with charge –q. At time t=0 we communicate to the particle velocity v_0, directed along the axes of the symmetry of the ring. Assume the ring cannot move. Discuss what would happen with the particle.*
Hint: the character of the motion will depend on the value of the initial velocity, of course.