

## Geometry.

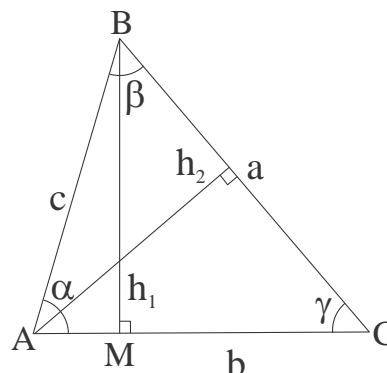
### Recap: Geometry of a triangle and trigonometry.

#### The Law of Sines.

$$c \sin \alpha = h_1 = a \sin \gamma \Rightarrow \frac{a}{\sin \alpha} = \frac{c}{\sin \gamma}$$

$$c \sin \beta = h_2 = b \sin \gamma \Rightarrow \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

The Law of Sines generalizes the fact that the greater side lies opposite to the greater angle.



#### The Extended Law of Sines

$$\sin \alpha = \frac{a}{2R} \Rightarrow \frac{a}{\sin \alpha} = 2R$$

$$\sin \beta = \frac{b}{2R} \Rightarrow \frac{b}{\sin \beta} = 2R$$

$$\sin \gamma = \frac{c}{2R} \Rightarrow \frac{c}{\sin \gamma} = 2R$$

The Law of Sines states that for any triangle ABC,

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R$$

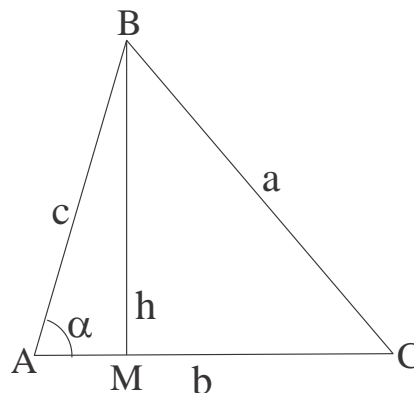
Where R is the radius of the circumscribed circle.

#### The Law of Cosines.

For any triangle ABC,

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$

To prove it, we consider right triangles formed by the height AM,



$$a^2 = h^2 + |MC|^2 ,$$

$$|MC| = b - |AM| = b - c \cos \alpha ,$$

$$h^2 = c^2 - |AM|^2 = c^2 - c^2(\cos \alpha)^2 ,$$

$$a^2 = c^2 - c^2(\cos \alpha)^2 + (b - c \cos \alpha)^2 =$$

$$= c^2 - c^2(\cos \alpha)^2 + b^2 - 2bc \cos \alpha + c^2(\cos \alpha)^2 = b^2 + c^2 - 2bc \cos \alpha$$

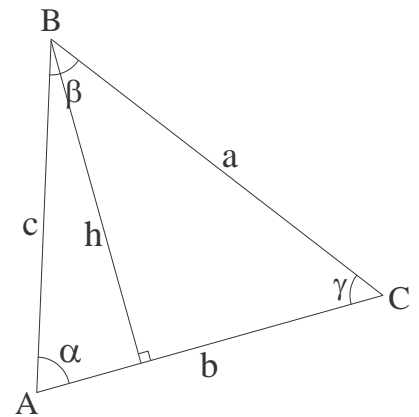
### Area of a triangle.

Using the Law of Sines in a standard formula, or simply considering the right triangle formed by an altitude opposite to vertex A, one obtains,

$$S_{\Delta ABC} = \frac{1}{2}hb = \frac{1}{2}bc \sin \alpha$$

Similarly, we also can get two more formulas:

$$S_{\Delta ABC} = \frac{1}{2}ab \sin \gamma = \frac{1}{2}ca \sin \beta$$



Using the Law of sines, we also have,

$$S_{\Delta ABC} = \frac{abc}{4R} = 2R^2 \sin \alpha \sin \beta \sin \gamma$$

where R is the radius of the circumscribed circle. We have also previously shown that

$$S_{\Delta ABC} = \frac{a + b + c}{2} r = sr$$

where r is the radius of the inscribed circle and s the semiperimeter. Finally, the area of the triangle can also be derived from the lengths of the sides by the Heron's formula,

$$S_{\Delta ABC} = \sqrt{s(s - a)(s - b)(s - c)}$$

## Trigonometry. Trigonometric formulae.

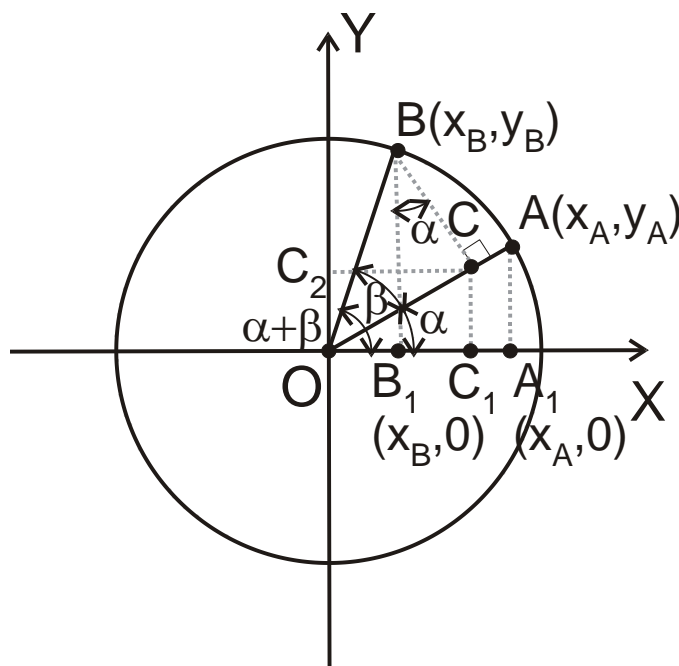
**Exercise.** Derive expressions for the sine and the cosine of the sum of two angles (see Figure),

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

**Solution.** Consider the circle of a unit radius,  $|\mathbf{OB}| = |\mathbf{OC}| = 1$ , in the Figure. Then,  $|\mathbf{OB}_1| = \cos(\alpha + \beta)$ ,  $|\mathbf{BB}_1| = \sin(\alpha + \beta)$ ,  $|\mathbf{OA}_1| = x_A = \cos \alpha$ ,  $|\mathbf{AA}_1| = y_A = \sin \alpha$ , etc.

Consequently,  $\sin(\alpha + \beta) = |\mathbf{BB}_1| = |\mathbf{CC}_1| + |\mathbf{BC}| \cos \widehat{\mathbf{CBB}}_1 = |\mathbf{OC}| \sin \alpha + |\mathbf{BC}| \cos \alpha = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ . Similarly,  $\cos(\alpha + \beta) = |\mathbf{OB}_1| = |\mathbf{OC}_1| - |\mathbf{B}_1\mathbf{C}_1| = |\mathbf{OC}| \cos \alpha - |\mathbf{BC}| \sin \alpha = \cos \alpha \cos \beta - \sin \alpha \sin \beta$ .



**Exercise.** Derive the addition formulas for sine and cosine using the figure of the triangle with an altitude drawn on the right.

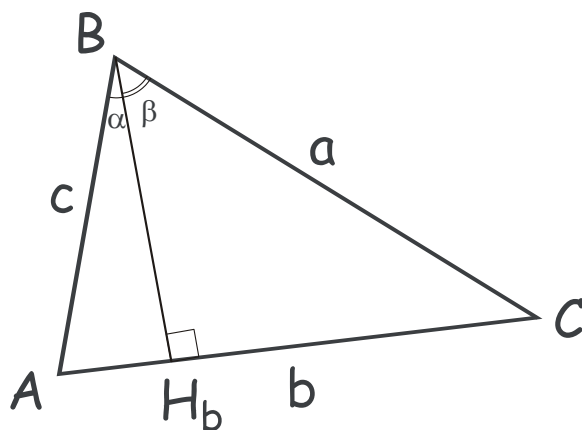
**Solution.**

1. Consider the area of the triangle ABC,

$$S_{ABC} = \frac{1}{2} ac \sin \widehat{ABC} = \frac{1}{2} ac \sin(\alpha + \beta)$$

On the other hand,

$$\begin{aligned} S_{ABC} &= S_{ABH_b} + S_{BCH_b} \\ &= \frac{1}{2} |AH_b| |BH_b| + \frac{1}{2} |CH_b| |BH_b| \\ &= \frac{1}{2} (c |BH_b| \sin \alpha + a |BH_b| \sin \beta) \end{aligned}$$



where  $|BH_b| = c \cos \alpha = a \cos \beta$ . Substituting this and combining the above equalities, we obtain  $\frac{1}{2} ac \sin(\alpha + \beta) = \frac{1}{2} ac (\sin \alpha \cos \beta + \cos \alpha \sin \beta)$ .

2. Now let us apply the cosines theorem to the triangle ABC,

$b^2 = a^2 + c^2 - 2ac \cos(\alpha + \beta)$ ,  $\cos(\alpha + \beta) = \frac{a^2 + c^2 - b^2}{2ac}$ , where  $b^2 = (c \sin \alpha + a \sin \beta)^2 = c^2 \sin^2 \alpha + a^2 \sin^2 \beta + 2ac \sin \alpha \sin \beta$ . Combining the two expressions, we obtain,

$$\begin{aligned}\cos(\alpha + \beta) &= \frac{a^2(1 - \sin^2 \beta) + c^2(1 - \sin^2 \alpha) - 2ac \sin \alpha \sin \beta}{2ac} \\ &= \frac{a \cos^2 \beta}{2c} + \frac{c \cos^2 \alpha}{2a} - \sin \alpha \sin \beta\end{aligned}$$

By using  $|BH_b| = c \cos \alpha = a \cos \beta$ ,  $\frac{a}{c} = \frac{\cos \alpha}{\cos \beta}$ , we obtain,

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

**Exercise.** Using the expression for the cosine of the sum of two angles derived above derive the expressions for the sine of the sum of two angles.

**Solution.** Using the formula for the sine and cosine of the supplementary angle,  $\sin\left(\alpha + \frac{\pi}{2}\right) = \cos \alpha$ ,  $\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin \alpha$ , and the above expression for  $\cos(\alpha + \beta)$  for we obtain,

$$\begin{aligned}\sin(\alpha + \beta) &= -\cos\left(\alpha + \beta + \frac{\pi}{2}\right) = -\left(\cos \alpha \cos\left(\beta + \frac{\pi}{2}\right) - \sin \alpha \sin\left(\beta + \frac{\pi}{2}\right)\right) \\ &= -\cos \alpha \cdot (-\sin \beta) + \sin \alpha \cos \beta = \sin \alpha \cos \beta + \cos \alpha \sin \beta\end{aligned}$$

**Exercise.** Using the expressions for the sine and the cosine of the sum of two angles derived above, derive expressions for,

1.  $\sin 2\alpha$
2.  $\cos 2\alpha$
3.  $\sin 3\alpha$
4.  $\cos 3\alpha$
5.  $\tan(\alpha \pm \beta)$
6.  $\cot(\alpha \pm \beta)$
7.  $\tan(2\alpha)$
8.  $\cot(2\alpha)$