Homework for February 11, 2017.

Algebra.

Review the classwork handout. Review the classwork and complete the exercises which were not solved in class. Try solving the unsolved problems from the previous homework (some are repeated below) and the following new problems.

1. Construct a proof that the set of all real numbers, $\mathbb{R}$, is uncountable, using the binary notation for real numbers.
2. Show that for the set of natural numbers, $\mathbb{N}$, cardinality of the set of all possible subsets is equal to that of a continuum of real numbers (Hint: use the binary number system).
3. Show that the set of points on any segment, $[a, b]$, on a line, has the same cardinality as
    a. the set of points on any other segment, $[c, d]$
    b. the set of points on a circle of unit radius
    c. the set of all points on a plane
    d. the set of all points in an n −dimensional hyper-cube
4. Show that each of the following sets has the same cardinality as a closed interval [0; 1] (i.e., there exists a bijection between each of these sets and [0; 1]).
    a. Interval [0; 1) [Hint: interval [0; 1] can be written as a union of two subsets, $A \cup B$, where A is a countable set including the interval end(s)].
    b. Open interval (0; 1)
    c. Set of all infinite sequences of 0s and 1s
    d. $\mathbb{R}$
    e. $[0,1] \times [0,1]$
5. Prove the following properties of countable sets. For any two countable sets, A, B,
    a. Union, $A \cup B$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
    b. Product, $A \times B = \{(a, b), a \in A, b \in B\}$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$

c. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 \ldots \cup A_n) = \aleph_0$
d. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the Cartesian product is also countable, $c(A_1 \times A_2 \ldots \times A_n) = \aleph_0$

6. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
   a. $1/15$
   b. $1/14$
   c. $1/7$
   d. $1/6$
   e. $0.33333\ldots = 0.(3)$
   f. $0.13333\ldots = 0.1(3)$

Geometry.

Review the classwork handout. Solve the unsolved problems from the previous homework. Solve the following problems.

Problems.

1. Review derivation of the equation describing an ellipse and derive in a similar way,
   a. Equation of an ellipse, defined as the locus of points P for which the distance to a given point (focus F_2) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.
   b. Equation of a hyperbola, defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$.
2. Consider all possible configurations of the Apollonius problem (i.e. different possible choices of circles, points and lines). How many possibilities are there? Make the corresponding drawings and write the equations for finding the Apollonius circle in one of them (of your choice).
3. Find the equation of the locus of points equidistant from two lines, $y = ax + b$ and $y = mx + n$, where a, b, m, n are real numbers.
4. Using the method of coordinates, prove that the geometric locus of points from which the distances to two given points have a given ratio, $q \neq 1$, is a circle.