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Algebra.

<u>Comparing finite and infinite sets. Cardinality.</u>

We have encountered different types of numbers, which can form finite sets, (set of 10 decimal digits, set of integers 1 to 100, etc.), or infinite sets, such as natural numbers, $\mathbb{N}$, integers, $\mathbb{Z}$, rational numbers, $\mathbb{Q}$, real numbers, $\mathbb{R}$. The basic idea of numbering the elements in a set leads us to the concept of comparing different sets. The numbering procedure amounts to establishing a bijection between the elements of a set, A, and the elements of a subset of natural numbers, $N \subseteq \mathbb{N}$. The obvious question then, can every set be numbered? Can set of all integers, $\mathbb{Z}$, be numbered? Of rational numbers, $\mathbb{Q}$? Of real numbers, $\mathbb{R}$? Obviously a set, A, can be numbered, if and only if, a bijection exists between this set and $\mathbb{N}$, $A \leftrightarrow \mathbb{N}$. Such set is said to be countable. Otherwise, the set is uncountable.

Two finite sets, A and B, such that a bijection exists $A \leftrightarrow B$ necessarily have equal number of elements, i.e. are equinumerous. Such sets form an **equivalence class**, corresponding to the natural number that denotes the number of elements in the sets of this class. Thus, natural numbers arise as a characteristic of equivalence classes of finite sets having the same number of elements. Georg Cantor, the originator of set theory, in 1874–1884 extended this concept to infinite sets, such as all integers, or real numbers, which led to comparing different types of infinite numbers, which are called **cardinals** (transfinite numbers). For a finite set A the cardinal, $c(A)$, is simply a number of elements, while for the infinite sets it has many properties reminiscent of it.

One can define addition and multiplication for cardinals, such that the commutative, associative and distributive laws hold, $c(A + B) = c(A) + c(B) = c(B + A)$, $c\big((A + B) + C\big) = c\big(A + (B + C)\big) = c(A) + c(B) + c(C)$, … .

An interesting and important property of cardinals, which was noted by Galileo in 1638, is that a cardinal of a subset of a given set can be equal to the cardinal of the set itself. The whole is not necessarily greater than its part, it can be equal to it, if "greater" means more numerous, and equal – equinumerous. For example, function $f(n) = 2n$ establishes a bijection between all integers and even integers, $f(n) = n^2$ establishes a bijection between the set of natural numbers and the subset of perfect squares, etc. All countable sets have the same cardinal as the set of natural numbers, $\aleph_0$. Note that using the addition rules defined above for the set of natural numbers, $\mathbb{N} = \mathbb{N}_{2k} \cup \mathbb{N}_{2k+1}$, where $\mathbb{N}_{2k}$ and $\mathbb{N}_{2k+1}$ are subsets of even and odd natural numbers, respectively, we obtain a seemingly paradoxical result,

$c(\mathbb{N}) = c(\mathbb{N}_{2k}) + (\mathbb{N}_{2k+1})$, or, $\aleph_0 = \aleph_0 + \aleph_0$,

Doubling $\aleph_0$ does not change it! Also note that we cannot deduce from the above that $\aleph_0 = 0$, because we do not know how to subtract cardinals.

Comparing cardinalities. If there exists an injection, $A \to B$, i. e. set A can be paired with a subset of set B, then $c(A) \leq c(B)$. This order relation on cardinalities has following useful properties,

- $c(A) \leq c(A)$
- $\{(c(A) \leq c(B)) \wedge (c(B) \leq c(C))\} \Rightarrow (c(A) \leq c(C))$
- $\{(c(A) \leq c(B)) \wedge (c(A) \leq c(B))\} \Rightarrow (c(A) = c(B))$

Countable sets. The following properties of the countable sets can be easily proven. For any two countable sets, A, B,

- Union, $A \cup B$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
- Product, $A \times B = \{(a, b), a \in A, b \in B\}$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
- For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 \ldots \cup A_n) = \aleph_0$

The examples of countable sets are,

- Set $\mathbb{Z}$ of all integers is countable, $c(\mathbb{Z}) = \aleph_0$
- Set $\mathbb{N} \times \mathbb{N}$ of pairs of positive integers is countable, $c(\mathbb{N} \times \mathbb{N}) = \aleph_0$
- Set $\mathbb{Q}$ of rational numbers is countable, $c(\mathbb{Q}) = \aleph_0$
- Set $\mathbb{Q} \times \mathbb{Q}$ of pairs of rational numbers is countable, $c(\mathbb{Q} \times \mathbb{Q}) = \aleph_0$
- Set of all polynomials with rational coefficients is countable

Uncountable sets. Continuum. The set of all real numbers, $\mathbb{R}$, is uncountable. An ingenious indirect proof of this was given by Cantor. The proof proceeds by contradiction. We assume that there exists a bijection between $\mathbb{N}$ and all real numbers, which can be written in the decimal form, $A.a_1a_2a_3\ldots$. We then construct a number that does not occur in the assumed denumeration sequence,

1. $A.a_1a_2a_3\ldots$
2. $B.b_1b_2b_3\ldots$
3. $C.c_1c_2c_3\ldots$
4. $D.d_1d_2d_3\ldots$

...

To do so, we consider a number $z = 0.z_1z_2z_3\ldots$, where z_1 is different from a_1, and is neither 0 or 9, z_2 is different from a_2, and is neither 0 or 9, and so on. z is a real number, but it is not included in the assumed denumeration above. Thus, we arrived at a contradiction, because we have assumed that all real numbers were included in the denumeration. This assumption must be false if there does exist a number which has been left out. Consequently, the assumption that a denumeration of the set of real numbers is possible is untenable, and therefore the opposite statement is true, i.e. that the set of real numbers is not countable. The cardinality of the set of real numbers is called continuum.

Whether there exists a set with a cardinal number greater than that of the set of integers, $\aleph_0$, but smaller than that of the set of real numbers, continuum, is a question which cannot be answered, i.e. such existence cannot be proved, or disproved. The assumption that such set exists, constitutes Hypothesis of the Continuum.

Cantor has shown that sets with greater and greater cardinal numbers exist, so there is no greatest cardinal number.

Theorem. Given set A, it is possible to construct set B with greater cardinal number, $c(B) > c(A)$.

An indirect proof of this theorem given by Cantor proceeds by considering a set of all possible subsets of A, $S = \{S_A\}, \forall S_A \subseteq A = \{a\}$. The set S includes both A and an empty set, $\emptyset$, and therefore $c(S) \geq c(A)$. Then, we assume that S has the same cardinality as A, i. e. that a bijection exists between S and A, $S \leftrightarrow A$, and arrive at a contradiction by constructing a subset of A, which is not an element of S. The bijection $S \leftrightarrow A$ "counts" all possible subsets of a set A by using the elements, $a \in A$, of the set itself, thus establishing a correspondence, $a \leftrightarrow S_a, \forall S_a \in S$. That this is not possible for finite sets, is rather obvious.

Exercise. Show that for a set of n elements, the set of all possible subsets has 2^n elements (Hint: remember Newton's binomial?).

In order to arrive at a contradiction, we consider a subset, $\{a'\} = A' \subseteq A$, which is composed of all such elements $a' \in A$, which do not belong to the corresponding subset $S_{a'}$ in the bijection, $a' \leftrightarrow S_{a'}$; $a' \notin S_{a'}$. This subset differs from any subset $S_{a'}$ of set A by at least one element, a', and therefore the assumption that such a bijection exists is untenable (remember, all possible subsets of A are included in the bijection, $a' \leftrightarrow S_{a'}$). It then follows that $c(S) > c(A)$, so for any set A the set of its all possible subsets has greater cardinal number than the set itself, and therefore there is no greatest cardinal number.

Exercise. Show that for the set of natural numbers, $\mathbb{N}$, cardinality of the set of all possible subsets is equal to that of a continuum of real numbers (Hint: use the binary number system).

Continuum and Dimensionality. The set of all real numbers, $\mathbb{R}$, which is uncountable, can be represented by points on a line, the coordinate axis.

Exercise. Show that the set of all real numbers from 0 to 1, $\{x\}, x \in \mathbb{R}, 0 \leq x \leq 1$, has the same cardinality (continuum) as the set of all real numbers, $\mathbb{R}$, i.e that a segment, $[0,1]$, is equivalent to an infinite line. Similarly, any segment, $[a, b]$, on a line, is equivalent to any other segment, $[c, d]$.

One might think that cardinality of a two-dimensional set of points, such as all points of the square with side 1, or all points on the plane, is greater than that of a one-dimensional continuum. It appears that this is not the case!

Theorem. The cardinal number, c_2, of the set of points in a square is the same as the cardinal number, c_1, of the set of points on a line segment.

Proof. It is sufficient to prove this equivalence for a square with side 1 and a segment $[0,1]$. Indeed, we can establish a correspondence between any point with coordinates $(x, y), x \in [0,1], y \in [0,1]$ in a square, and a point $z \in [0,1]$ on a segment in the following way. Let us write numbers $x = 0.x_1x_2x_3 \ldots$ and $y = 0.y_1y_2y_3 \ldots$ in decimal notation, where we identify numbers ending in an infinite sequence of 9's, which represent rational numbers, with finite-length decimals, $a.bc999(9) = a.bc$, so infinite sequences of 9 do not appear. We then assign a number $z = 0.x_1y_1x_2y_2x_3y_3 \ldots, z \in [0,1]$ to the point (x, y). To every point in a square we have thus assigned a unique number on a segment $[0,1]$, since for any other point, (x', y'), either x', or y', or both, differ from x and y, respectively, in at least one digit. Therefore, the corresponding number $z' \in [0,1]$ will also be different. Note, that the above described correspondence is not a bijection, but an injection, because numbers of the type $z = a.b \ldots 0909(09) \ldots, z \in [0,1]$, which do correspond to points on a segment $[0,1]$ have no corresponding points on a square. While it is possible to modify this correspondence so that it becomes a bijection, this is not necessary for our purposes, as the existence of an injection $[0,1] \times [0,1] \rightarrow [0,1]$ already proves that $c_2 \leq c_1$. The obvious surjection established by associating just one side of a square with the segment shows that $c_2 \geq c_1$. It then follows that $c_2 = c_1$. One can extend this argument to show that the cardinal number of a set of points in a cube is also equal to the cardinal number of the set of points on a segment, $c_3 = c_1$.

Exercise. Show that the cardinal number of an n −dimensional hyper-cube is equal to the cardinal number of a segment, $c_n = c_1$.

The main conclusion from the above observations is that the dimension of a set of points depends not only on the cardinal number of the set. While the fact that the cardinality of a square, or cube, is equal to that of a segment seems to disagree with the intuitive notion of dimensionality, the fundamental reason that the above correspondence works is that it is not continuously mapping one set to the other. In fact, if we vary point z on a segment continuously from 0 to 1, the corresponding points in a square would appear in a completely random and discontinuous manner. This is the subject that is studied in topology.

How then does the set of all rational points on the segment $[0,1]$ compare to the set of all points on this segment? We have already proven that the set of rational points is countable and has the cardinal $\aleph_0$, while the set of all points on the segment $[0,1]$ is an uncountable continuum. The following theorem provides an alternative proof of this fact.

Theorem. The cardinal number of a denumerable set of points on a segment is less than that of an arbitrarily small, of length $\varepsilon \le 1$, part of that segment.

Proof. It is sufficient to prove this for a segment $[0,1]$. Let us arrange all points of the countable set $A = \{a_n\}, n \in \mathbb{N}, A \subset [0,1]$, in a sequence, $a_1, a_2 a_3, \dots, a_n, \dots$... Let us now enclose each point in a segment, such that the length of the segment enclosing n-th point is ${}^{\varepsilon}/_{10^n}$. While some of these segments might be overlapping, the total length covered by these segments is not larger than the total length of the segments, $l = \frac{\varepsilon}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^n} + \cdots\right) = \frac{\varepsilon}{10}\frac{1}{1-\frac{1}{10}} = \frac{\varepsilon}{9}$. Because ε can be arbitrarily small, in the language of measure theory, the denumerable set of points has measure zero.