Homework for January 28, 2018.

Algebra.

Review the classwork handout. Review and solve the classwork exercises which were not solved (some are repeated below). Solve the following problems.

1. Prove the following properties of the Cartesian product,
   a. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   b. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   c. $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$
2. Find the Cartesian product, $A \times B$, of the following sets,
   a. $A = \{a, b\}, B = \{\uparrow, \downarrow\}$
   b. $A = \{June, July, August\}, B = \{1,15\}$
   c. $A = \emptyset, B = \{1,2,3,4,5,6,7,8,9\}$
3. Describe the set of points determined by the Cartesian product, $A \times B$, of the following sets (illustrate schematically on a graph),
   a. $A = [0,1], B = [0,1]$ (two segments from 0 to 1)
   b. $A = [-1,1], B = (-\infty, \infty)$
   c. $A = (-\infty, 0], B = [0, \infty)$
   d. $A = (-\infty, \infty), B = (-\infty, \infty)$
   e. $A = [0,1), B = \mathbb{Z}$ (set of all integers)
4. Propose 3 meaningful examples of a Cartesian product of two sets.
5. $n_A = |A|$ is the number of elements in a set A.
   a. What is the number of elements in a set $A \times A$
   b. What is the number of elements in a set $A \times (A \times A)$
6. Present examples of binary relations that are, and that are not equivalence relations.
7. For each of the following relations, check whether it is an equivalence relation and describe all equivalence classes.
   a. On $\mathbb{R}$: relation given by $x \sim y$ if $|x| = |y|$
   b. On $\mathbb{Z}$: relation given by $a \sim b$ if $a \equiv b \bmod 5$
   c. On $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 + x_2 = y_1 + y_2$; describe the equivalence class of (1, 2)
   d. Let $\sim$ be the relation on the set of all directed segments in the plane defined by $\overrightarrow{A\,B} \sim \overrightarrow{A'\,B'}$ if $ABB'A'$ is a parallelogram.

e. On the set of pairs of integers, $\{(a, b), a, b \in \mathbb{Z}, b \neq 0\}$, $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 b_2 = a_2 b_1$. Describe these equivalence classes. Is the set of the obtained equivalence classes countable?

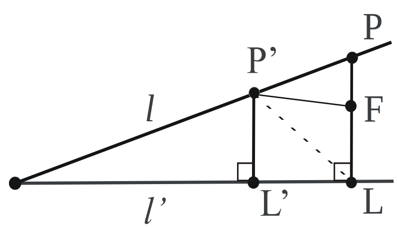
8. Let $f: X \xrightarrow{f} Y$ be a function. Define a relation on X by $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Prove that it is an equivalence relation. Describe the equivalence classes for the equivalences defined by the following functions on $\mathbb{R}$.
   a. $f(x) = x^2$: $x \sim y$ if $x^2 = y^2$.
   b. $f(x) = \sin x$: $x \sim y$ if $\sin x = \sin y$.

Geometry.

Review the previous classwork notes. Solve the following problems from the last homework (if you have not solved them yet).

Problems.

1. Review derivation of the equation describing an ellipse. In a similar way, try deriving,
   a. Equation of an ellipse, defined as the locus of points P for which the distance to a given point (focus F_2) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.
   b. Equation of a hyperbola, defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$.
2. Given two lines, l and l', and a point F not on any of those lines, find point P on l such that the (signed) difference of distances from it to l' and F, $|P'L'| - |P'F|$, is maximal. As seen in the figure, for any P' on l the distance to l', $|P'L'| \leq |P'L| \leq |P'F| + |FL|$, where $|FL|$ is the distance from F to l'. Hence, $|P'L'| - |P'F| \leq |FL|$, and the difference is largest ($= |FL|$) when point P belongs to the perpendicular FL from point F to l'.



3. Given line l and points F_1 and F_2 lying on different sides of it, find point P on the line l such that the absolute value of the difference in distances from P to points F_1 and F_2 is maximal. As above, let F_2' be the reflection of F_2 in l. Then for any point X on l, $|XF_2| - |XF_1'| \leq |F_1F_2'|$