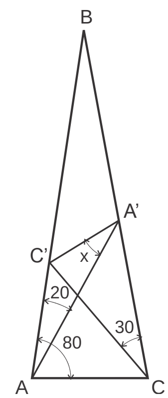
January 14, 2018

Geometry.

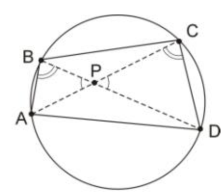
Homework problems review.

1. In an isosceles triangle ABC with the angles at the base, $\angle BAC = \angle BCA = 80°$, two Cevians CC' and AA' are drawn at an angles $\angle BCC' = 30°$ and $\angle BAA' = 20°$ to the sides, CB and AB, respectively (see Figure). Find the angle $\angle AA'C' = x$ between the Cevian AA' and the segment $A'C'$ connecting the endpoints of these two Cevians.



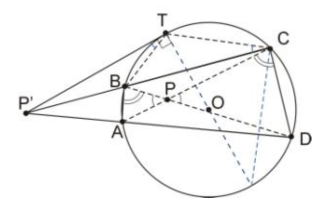
2. Write the proof of the Euclid theorem, which states the following. If two chords AD and BC intersect at a point P' outside the circle, then

$$|P'A||P'D| = |P'B||P'C| = |PT|^2 = d^2 - R^2,$$



where $|PT|$ is a segment tangent to the circle (see Figure).

3. Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,



$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle (try using the triangle inequality).

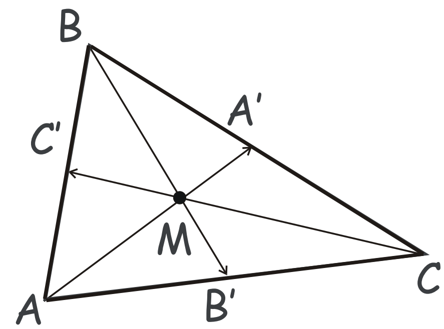
4. Using the Ptolemy's theorem, prove the following:
   a. Given an equilateral triangle $\Delta\, ABC$ inscribed in a circle and a point Q on the circle, the distance from point Q to the most distant vertex of the triangle is the sum of the distances from the point to the two nearer vertices.
   b. In a regular pentagon, the ratio of the length of a diagonal to the length of a side is the golden ratio, ϕ.
5. Given a circle of radius R, find the length of the sagitta (Latin for arrow) of the arc AB, which is the perpendicular distance CD from the arc's midpoint (C) to the chord AB across it.

6. Prove the Viviani's theorem:

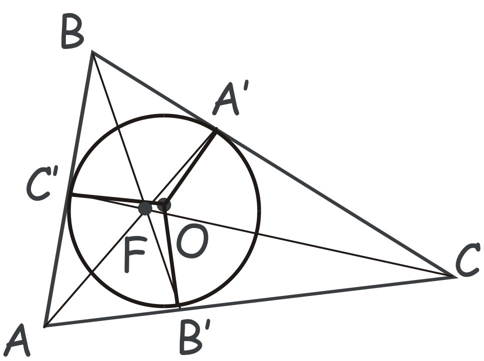
   The sum of distances of a point P inside an equilateral triangle or on one of its sides, from the sides, equals the length of its altitude. Or, alternately,

   From a point P inside (or on a side) of an equilateral triangle ABC drop perpendiculars PP_a, PP_b, PP_c to its sides. The sum $|PP_a| + |PP_b| + |PP_c|$ is independent of P and is equal to any of the triangle's altitudes.

7. *Three Points are taken at random on an infinite plane. Find the chance of their being the vertices of an obtuse-angled Triangle. Hint: use the Viviani's theorem.
8. In a triangle ABC, Cevian segments AA', BB' and CC' are concurrent and cross at a point M (point C' is on the side AB, point B' is on the side AC, and point A' is on the side BC). Given the ratios $\frac{AC'}{C'B} = p$ and $\frac{AB'}{B'C} = q$, find the ratio $\frac{AM}{MA'}$ (express it through p and q).



9. What is the ratio of the two segments into which a line passing through the vertex A and the middle of the median BB' of the triangle ABC divides the median CC'?
10. In a triangle ABC, A', B' and C' are the tangent points of the inscribed circle and the sides BC, AC, and AB, respectively (see Figure). Prove that cevians AA', BB' and CC' are concurrent (their common point F is called the Gergonne point).



11. The expression $d^2 - R^2$ is called the power of point P with respect to a circle of radius R, if $d = |PO|$ is the distance from P to the center O of the circle. The power is positive for points outside the circle; it is negative for points inside the circle, and zero on the circle.
    a. What is the smallest possible value of the power that a point can have with respect to a given circle of radius R? Which point is that?
    b. What is the locus of all points of constant power p (greater than the above minimum) with respect to a given circle? Write the relation between the (x, y) coordinates of such points, assuming that circle is centered at the origin $(0,0)$.
    c. Let t^2 be the power of point P with respect to a circle R. What is the geometrical meaning of it?

12. Review the solution of the radical axis of two circles problem: find the locus of points whose powers with respect to two non-concentric circles are equal. Consider situation when circles are concentric.
13. Complete the following exercises from class. Find the locus of points satisfying each of the following equations or inequalities (graph it on a coordinate plane).
    a. $|x| = |y|$
    b. $|x| + x = |y| + y$
    c. $|x|/x = |y|/y$
    d. $[y] = [x]$
    e. $\{y\} = \{x\}$
    f. $x^2 - y^2 \geq 0$
    g. $x^2 + y^2 \leq 1$
    h. $x^2 + 8x = 9 - y^2$
14. Describe the locus of all points (x, y) equidistant to the X-axis (i. e. the line $y = 0$) and a given point P $(0,2)$ on the Y-axis. Write the formula relating y and x for these points.
15. (Skanavi 15.105) Find the (x, y) coordinates of the vertex C of an equilateral triangle ABC if A and B have coordinates $A(1,3)$ and $B(3,1)$, respectively.
16. (Skanavi 15.106) Find the (x, y) coordinates of the vertices C and D of a square $ABCD$ if A and B have coordinates $A(2,1)$ and $B(4,0)$, respectively.
17. *Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.