January 14, 2018

Algebra.

Recap: Elements of Set Theory.

Arrangements and Derangements.

Arrangements of a subset of k distinct objects chosen from a set of n distinct objects are $A_n^k = \frac{n!}{(n-k)!}$ permutations [order matters] of distinct subsets of k elements chosen from that set. The total **number of arrangements** of all subsets of a set of n distinct objects is the number of unique sequences [order matters] that can be formed from any subset of $0 \leq k \leq n$ objects of the set,

$$a_n = \sum_{k=0}^{n} A_n^k = \sum_{k=0}^{n} k! \binom{n}{k} = \sum_{k=0}^{n} \frac{n!}{(n-k)!} = n! \sum_{k=0}^{n} \frac{1}{k!} \equiv ¡n$$

This number is obviously larger than the number of permutations of n distinct objects given by $n!$. Hence, a supfactorial, $¡n$, notation has been suggested. It is easy to check that $¡n$ satisfies the following recurrence relation,

$$¡n = n \cdot ¡(n-1) + 1$$

For very large $n \gg 1$, the supfactorial is nearly a constant times the factorial, $¡n \approx e \cdot n!$

Exercise. How many possible passwords can be composed using an alphabet of $n = 26$ letters, if a password is required to have at least 8 characters and have no repeating characters? Answer: $a_{26} - a_7 = 26! \sum_{k=8}^{26} \frac{1}{k!}$.

A (complete) **derangement** is a permutation of the elements of a set of distinct objects such that none of the objects appear in their original position. The number of derangements of a set of n distinct objects (or permutations of n distinct objects with no rencontres, or permutations with no fixed point) is smaller than $n!$ and is called the subfactorial, $!n$. It can be obtained by using the inclusion-exclusion principle. The universal set of permutations P has $n!$ elements. Denote P_1 the subset of permutations that keep element 1 in its place, P_2 those that keep element 2 in its place, P_k that keep element k in its place, and so on. The set of permutations that keep at least 1 element in its original place is then, $P_{>1} = P_1 \cup P_2 \cup P_3 \dots \cup P_n$. The number of derangement is given by the number of elements complementing this set to P,

$$d_n = \left| P - \sum_{i=1}^{n} P_i \right| =$$

$$n! - \sum_{i=1}^{n} |P_i| + \sum_{1 \le i \le j \le n}^{n} \left|P_i P_j\right| - \sum_{1 \le i \le j \le k \le n}^{n} \left|P_i P_j P_k\right| \dots + (-1)^n |P_1 \dots P_n|$$

Using the fact that $|P_i|, \left|P_i P_j\right|, \left|P_i P_j P_l\right|, \dots$ are equal to $(n-1)!, (n-2)!, (n-3)!, \dots$, correspondingly, for every choice of $i, \{i, j\}, \{i, j, l\}, \dots$ and there are $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \dots \binom{n}{k}, \dots$ such choices, respectively, we obtain,

$$d_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n \binom{n}{n} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \equiv\, !n$$

The number of derangements also obeys the following recursion relations,

$d_n = n \cdot d_{n-1} + (-1)^n$, or, $!n = n \cdot !(n-1) + (-1)^n$, and,

$d_n = (n-1) \cdot (d_{n-1} + d_{n-2})$, or, $!n = (n-1) \cdot \big(!(n-1) + !(n-2)\big)$.

Note that the latter recursion formula also holds for $n!$; for very large $n \gg 1$, the subfactorial is nearly a factorial divided by a constant, $!n \approx \frac{n!}{e}$. Starting with $n = 0$, the numbers of derangements of n are,

1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570, 176214841, 2290792932, ...

Exercise. A group of n men enter a restaurant and check their hats. The hat-checker is absent minded, and upon leaving, redistributes the hats back to the men at random. What is the probability, P_n, that no man gets his correct hat?

This is the old hats problem, which goes by many names. It was originally proposed by French mathematician P. R. de Montmort in 1708, and solved by him in 1713. At about the same time it was also solved by Nicholas Bernoulli using inclusion-exclusion principle.

An alternative solution is to devise a recurrence by noting that for a full derangement, every of n men should get somebody else's hat. Assume man x got the hat of man y. Assuming that man y got the hat of man x, there are d_{n-2} such possible derangements. However, we also have to account for the possibility that man y, whose hat went to man x, did not get "his" hat of man x in return. This gets us to the situation of the full derangement for $n-1$ men. Adding the two possibilities and multiplying with $n-1$ possible choices of man y we obtain, $d_n = (n-1)(d_{n-1} + d_{n-2})$. For the probability we then obtain, or, $P_n = \frac{d_n}{n!} = P_{n-1} - \frac{1}{n}(P_{n-1} - P_{n-2})$, wherefrom the expression for the derangement probability can be derived by setting up a "telescoping" sum.

If some, but not necessarily all, of the items are not in their original ordered positions, the configuration can be referred to as a partial derangement. The number of partial derangements with k fixed points (rencontres) is,

$$d_{n,k} = \binom{n}{k} d_{n-k} = \binom{n}{k} \sum_{p=0}^{k} \frac{(-1)^p}{p!}$$

Here is the beginning of this array.

n/k	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	1							
1	0	1						
2	1	0	1					
3	2	3	0	1				
4	9	8	6	0	1			
5	44	45	20	10	0	1		
6	265	264	135	40	15	0	1	
7	1854	1855	924	315	70	21	0	1

Some homework problems.

Exercise. Verify the following recurrences for the number of arrangements, $a_n = n!\ \sum_{k=0}^{n} \frac{1}{k!}$ ≡¡ n,

$$a_n = n!\ \sum_{k=0}^{n} \frac{1}{k!} = n!\ \sum_{k=0}^{n-1} \frac{1}{k!} + \frac{n!}{n!} = n \cdot a_{n-1} + 1$$

Solution.

$$a_n = n!\ \sum_{k=0}^{n} \frac{1}{k!} = n!\ \sum_{k=0}^{n-1} \frac{1}{k!} + 1 = n \cdot (n-1)!\ \sum_{k=0}^{n-1} \frac{1}{k!} + 1 =$$
$$(n-1+1) \cdot (n-1)!\ \sum_{k=0}^{n-1} \frac{1}{k!} + 1 = (n-1) \cdot (n-1)! \cdot \sum_{k=0}^{n-1} \frac{1}{k!} +$$
$$(n-1)! \cdot \left(\sum_{k=0}^{n-2} \frac{1}{k!} + \frac{1}{(n-1)!} \right) + 1 = (n-1) \cdot (a_{n-1} + a_{n-2}) + 2$$

Homework for January 7, 2018.

1. Using the inclusion-exclusion principle, find how many natural numbers $n < 100$ are not divisible by 3, 5 or 7.

Solution. For $n < 100$, there are 33 divisible by 3, $|A_3| = 33$, 19 divisible by 5, $|A_5| = 19$, 14 numbers divisible by 7, $|A_7| = 14$. Also, there are 6 numbers divisible by $3 \cdot 5 = 15$, 4 divisible by $3 \cdot 7 = 21$, 2 divisible by $5 \cdot 7 = 35$, and none divisible by $3 \cdot 5 \cdot 7 = 105$. Hence, the answer is $99 - |A_3 + A_5 + A_7| = 99 - |A_3| - |A_5| - |A_7| + |A_{3 \cdot 5}| - |A_{3 \cdot 7}| - |A_{5 \cdot 7}| = 99 - (33 + 19 + 14 - 6 - 4 - 2 + 0) = 99 - 54 = 45$.

2. Four letters a, b, c, d, are written down in random order. Using the inclusion-exclusion principle, find probability that at least one letter will occupy its alphabetically ordered place? What is the probability for five letters?
3. Using the inclusion-exclusion principle, find the probability that if we randomly write a row of digits from 0 to 9, no digit will appear in its proper ordered position.
4. Secretary prepared 5 different letters to be sent to 5 different addresses. For each letter, she prepared an envelope with its correct address. If the 5 letters are to be put into the 5 envelopes at random, what is the probability that
   a. no letter will be put into the envelope with its correct address?

b. only 1 letter will be put into the envelope with its correct address?
c. only 2 letters will be put into the envelope with its correct address?
d. only 3 letters will be put into the envelope with its correct address?
e. only 4 letters will be put into the envelope with its correct address?
f. all 5 letters will be put into the envelope with its correct address?

5. Among 24 students in a class, 14 study mathematics, 10 study science, and 8 study French. Also, 6 study mathematics and science, 5 study mathematics and French, and 4 study science and French. We know that 3 students study all three subjects. How many of these students study none of the three subjects?
6. In a survey on the students' chewing gum preferences, it was found that
   a. 20 like juicy fruit.
   b. 25 like spearmint.
   c. 33 like watermelon.
   d. 12 like spearmint and juicy fruit.
   e. 16 like juicy fruit and watermelon.
   f. 20 like spearmint and watermelon.
   g. 5 like all three flavors.
   h. 4 like none.

   How many students were surveyed?

Homework for January 14, 2018.

1. Construct bijections between the following sets:
   a. (subsets of the set $\{1, \dots, n\}$) ↔ (sequences of zeros and ones of length n)
   b. (5-element subsets of $\{1, \dots, 15\}$)↔ (10-element subsets of $\{1, \dots, 15\}$)
   c. [set of all ways to put 10 books on two shelves (order on each shelf matters)]↔ (set of all ways of writing numbers 1, 2, . . . , 11 in some order) [Hint: use numbers 1. . . 10 for books and 11 to indicate where one shelf ends and the other begins.]
   d. (all integer numbers)↔ (all even integer numbers)
   e. (all positive integer numbers)↔ (all integer numbers)
   f. (interval (0,1))↔ (interval (0,5))
   g. (interval (0,1))↔ (halfline $(1, \infty)$) [Hint: try $1/x$.]

h. (interval (0,1))↔ (halfline $(0, \infty)$)
i. (all positive integer numbers)↔ (all integer numbers)

2. Let A be a finite set, with 10 elements. How many bijections f: $A \to A$ are there? What if A has n elements?
3. Let $f: \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = n^2$ Is this function injective? surjective?
4. Hotel Infinity is a fictional hotel with infinitely many rooms, numbered 1, 2, 3,. . . . Each hotel room is single occupancy: only one guest can stay there at any time.
   a. At some moment, Hotel Infinity is full: all rooms are occupied. Yet, when 2 more guests arrive, the hotel manager says he can give rooms to them, by moving some of the current guests around. Can you show how? (Hint: Construct a bijection between sets $\{-1, 0, 1, 2, \ldots\}$ and N).
   b. At some moment, Hotel Infinity is full: all rooms are occupied. Still, the management decides to close half of the rooms — all rooms with odd numbers — for renovation. They claim they can house all their guests in the remaining rooms. Can you show how? (Hint: Construct a bijection between the set of all even positive integers $\{2, 4, 6, \ldots\}$ and N).
   c. Next to Hotel infinity, a competitor has built Hotel Infinity 2, with infinitely many rooms num- bered by all integers: $\ldots, -2, -1, 0, 1, 2, \ldots$. Yet, the management of original Hotel Infinity claims that their hotel is no smaller than the competition: they could house all the guests of Hotel Infinity 2 in Hotel Infinity. Could you show how? (Hint: Construct a bijection between the set of all integer numbers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and N).
5. * If 9 dies are rolled, what is the probability that all 6 numbers appear?
6. * How many permutations of the 26 letters of English alphabet do not contain any of the words *pin*, *fork*, or *rope*?