November 19, 2017

Algebra.

Solutions to some homework problems.

1. **Problem.** Write the first few terms in the following sequence ($n \geq 1$),

$$n\ fractions \left\{ \begin{array}{l} \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} \\ \qquad \qquad \ldots + \cfrac{1}{1+x} \end{array} \right. = f_n$$

   a. Try guessing the general formula of this fraction for any n.

   b. Using mathematical induction, try proving the formula you guessed.

Solution. $n = 1$: $f_1 = \frac{1}{1+x}$; $n = 2$: $f_2 = \frac{1}{1+\frac{1}{1+x}} = \frac{1+x}{2+x}$; $n = 3$, $f_3 = \frac{1}{1+\frac{1}{1+\frac{1}{1+x}}} = \frac{2+x}{3+2x}$; $n = 4$, $f_4 = \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+x}}}} = \frac{3+2x}{5+3x}$; $f_5 = \frac{5+3x}{8+5x}$; … .

From the definition, we can write the recurrence, $f_{n+1} = \frac{1}{1+f_n}$. We note, that if $f_n = \frac{a_n+b_nx}{c_n+d_nx}$, then $f_{n+1} = \frac{c_n+d_nx}{(a_n+c_n)+(b_n+d_n)x}$. Hence, in each next term, f_{n+1}, in the sequence, the numerator is equal to the denominator of the previous term, f_n, while the numbers in the denominator are the sums of the corresponding numbers in the numerator and the denominator of the previous term, f_n, thus forming the Fibonacci sequence, $\{F_n\} = \{1,1,2,3,5,8,13,\ldots\}$. We can thus guess,

a. $n\ fractions$: $f_1 = \frac{1}{1+x}$, $f_n = \frac{F_n+F_{n-1}x}{F_{n+1}+F_nx}$, $n > 1$

b. <u>Base</u>: $f_2 = \frac{1+x}{1+2x}$

   <u>Induction</u>: Using the recurrence implied in the definition,

$$f_{n+1} = \frac{1}{1+f_n} = \frac{1}{1+\frac{F_n+F_{n-1}x}{F_{n+1}+F_nx}} = \frac{F_{n+1}+F_nx}{F_{n+1}+F_n+F_nx+F_{n-1}x} = \frac{F_{n+1}+F_nx}{F_{n+2}+F_{n+1}x}.$$

2. **Problem.** Can you prove that,
   a.
   $\frac{3+\sqrt{17}}{2} = 3 + \frac{2}{3+\frac{2}{3+\frac{2}{3+\cdots}}}$?
   b. $1 = 3 - \frac{2}{3-\frac{2}{3-\frac{2}{3-\cdots}}}$?
   c.
   $\frac{4}{2+\frac{4}{2+\frac{4}{2+\cdots}}} = 1 + \frac{1}{4+\frac{1}{4+\frac{1}{4+\cdots}}}$?

   Find these numbers?

   Solution. Consider a general continued fraction,

$$x = a + \frac{b}{a + \frac{b}{a + \frac{b}{a + \cdots}}}$$

If a number exists, which is equal to the above infinite continued fraction, then it must satisfy the equation, $x = a + \frac{b}{x} \Leftrightarrow x^2 - ax - b = 0$ $\Leftrightarrow x = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 + b}$. If a and b are positive, then x must also be positive, so $x = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 + b}$.

a. Following the above argument with $a = 3$, $b = 2$, we obtain,
   $x = \frac{3}{2} + \sqrt{\left(\frac{3}{2}\right)^2 + 2} = \frac{3+\sqrt{17}}{2}$
b. In this case, $a = 3$, but $b = -2$ is negative. Applying the above considerations naively, we obtain, $x = 3 - \frac{2}{x} \Leftrightarrow x^2 - 3x + 2 = 0$ $\Leftrightarrow (x-1)(x-2) = 0$, i.e. there are two equally "legitimate" answers, $x = 1$, or $x = 2$. What this means, is that assumption that there exist unique number encoded by the given infinite continued fraction is wrong: there exist no such number! In fact, this can also be understood by looking at finite truncations approximating this

continued fraction. If the continued fraction is truncated after subtracting 2 and before division by 3, then it is equal to 1,

$3-\frac{2}{3-2}=1, 3-\frac{2}{3-\frac{2}{3-2}}=1, \dots .$

If, on the other hand, the truncation is after division by 3 and before subtracting 2, then we obtain a sequence of numbers approaching 2,

$3-\frac{2}{3}=2\frac{1}{3}, 3-\frac{2}{3-\frac{2}{3}}=2\frac{1}{7}, 3-\frac{2}{3-\frac{2}{3-\frac{2}{3}}}=2\frac{1}{15}, \dots .$

c. Denote

$$x=\cfrac{4}{2+\cfrac{4}{2+\cfrac{4}{2+\cdots}}}=\frac{4}{2+x}$$

Then, $x^2+2x-4=0 \Leftrightarrow x=-1\pm\frac{\sqrt{5}}{2}$, and $x>0$. Hence, $x=-1+\frac{\sqrt{5}}{2}$.

Similarly, denote

$$y=\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\cdots}}}=\frac{1}{4+y}$$

Then, $y^2+4y-1=0 \Leftrightarrow y=-2\pm\frac{\sqrt{5}}{2}$, and $y>0$. Hence, $y=-2+\frac{\sqrt{5}}{2}$, and $1+y=-1+\frac{\sqrt{5}}{2}=x$.