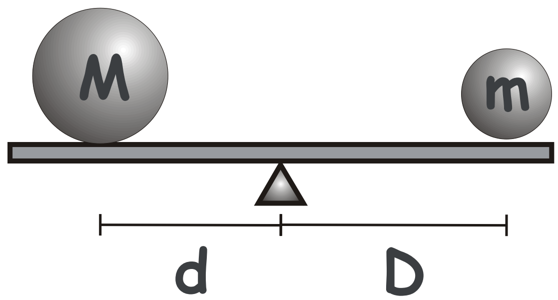
November 5, 2017

Geometry.

The Method of the Center of Mass (mass points): Solving problems using the Law of Lever (mass points). Menelaus theorem. Pappus theorem.



Theorem (Law of Lever). Masses (weights) balance at distances from the fulcrum, which are inversely proportional to their magnitudes,

$$\frac{D}{d} = \frac{M}{m} \Leftrightarrow Md = mD$$

For commensurate masses, $= p \cdot w, m = q \cdot w,\ \ p, q \in \mathbb{N}$, the Law was proven using the main "trick" of the mass points method: each of the two masses is split into $2p$ and $2q$ smaller masses, $w/2$, respectively, which are then re-positioned in pairs around the original masses so that positions of the center of mass (COM) for each of the two original masses do not change, but the COM position for the whole system becomes obvious.

In order to prove the Law of Lever for incommensurate masses, we first make the following observation.

Lemma. If two commensurate masses m and M are placed at distances D and d from the fulcrum, respectively, then M goes up if and only if $Md < mD$,

$$(M\ rises\ up) \Leftrightarrow (Md < mD)$$

First, if distances d and D are incommensurate, we move mass M slightly, to a position d' which is commensurate with D, but such that M still rises up. Therefore, we only need to consider case when d and D are commensurate.

Since M rises up, we need to increase mass M to achieve balance. Let $M' > M$ be such that M' and m balance. Using the Law of Lever for commensurate masses we have, $M' = m\frac{D}{d}$ (because distances are commensurate, so are the masses). Since $M < M' = m\frac{D}{d}$, it follows that $Md < mD$. Conversely, if $Md < mD$ we can increase it to $M' = m\frac{D}{d}$, which balances m. Decreasing mass from back to M will cause it to rise.

Corollary. The converse statement immediately follows via excluded middle,

$$(M\ goes\ down) \Leftrightarrow (Md > mD)$$

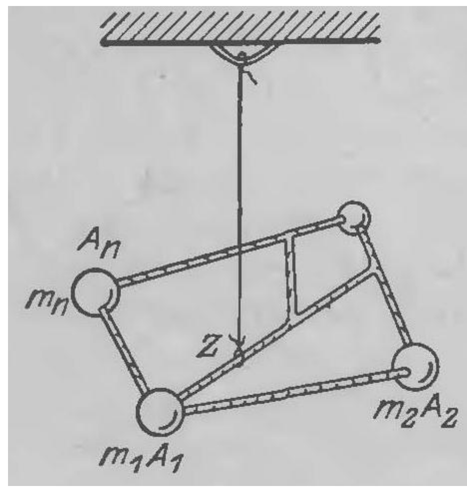
Proof (case of incommensurate masses). Let now two incommensurate masses m and M, be placed at distances d and D from the fulcrum, respectively, such that the Law of Lever is satisfied, $Md = mD$. Assume that the masses nevertheless do not balance, for example, M goes down. Decrease mass M by a small amount, turning it into M', such that it still goes down, but is now commensurate with m. Now m and M' are commensurate, and $mD > M'd$, which means that M' should rise. This contradicts our assumption, so m and M must balance. Note that in the above we used a non-trivial fact that a commensurate mass, or distance can be found that differs from the given incommensurate one by an arbitrarily small amount. This means that for any irrational number there exists a rational number, which differs from it as little as we want, i. e. that rational numbers are dense.

Solving problems using the Law of Lever.

For the objects in the uniform gravitational field, the Center of Gravity and the Center of Mass are equivalent. Archimedes uses the concept by considering bodies with the uniform density and defining the Center of Gravity based on postulated properties.

Heuristic Definitions of the Center of Mass (Center of Gravity) known to Greeks.

1. The point such that if suspended at it, an object will remain motionless in the equilibrium, independent of the position that it is placed.
2. The point common to all the lines passing through the point at which the object is suspended
3. The point common to all lines on which the object balances.



Archimedes' postulates on the properties of the Center of Gravity (COM).

1. The COM of similar figures are similarly situated.
2. The COM of a convex figure lies within the figure.
3. If an object is cut in two pieces, then its COM lies on the line joining the COM's of the pieces, and its position satisfies the Law of Lever.

However, the situation is much simpler if we only consider point masses.

Properties of the Center of Mass for a system of point masses.

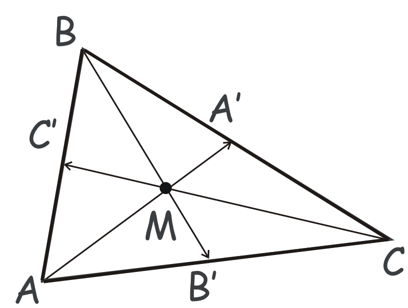
1. Every system of finite number of point masses has unique center of mass (COM).
2. For two point masses, m_1 and m_2, the COM belongs to the segment connecting these points; its position is determined by the Archimedes

lever rule: the point's mass times the distance from it to the COM is the same for both points, $m_1 d_1 = m_2 d_2$.

3. The position of the system's center of mass does not change if we move any subset of point masses in the system to the center of mass of this subset. In other words, we can replace any number of point masses with a single point mass, whose mass equals the sum of all these masses and which is positioned at their COM.

Solving problems using the COM.

Given a system of points and lines, one can derive various relations, such as concurrence of particular lines connecting some of the points, or the ratio of the lengths of different segments by associating certain masses with these points (i.e. placing point masses at their positions) and considering the center of mass of the obtained system of mass points.



Exercise. Prove that the medians of an arbitrary triangle ABC are concurrent (cross at the same point M).

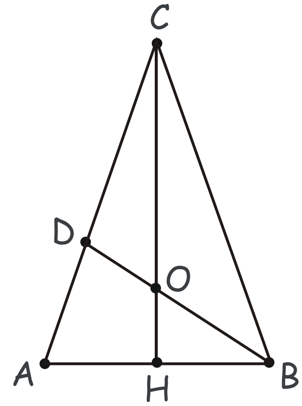
Exercise. Prove that the bisectors of an arbitrary triangle ABC are concurrent (cross at the same point O).

COM solutions of the selected homework problems.

1. **Problem.** Prove that medians of a triangle divide one another in the ratio 2:1, in other words, the medians of a triangle "trisect" one another (Coxeter, Gretzer, p.8).

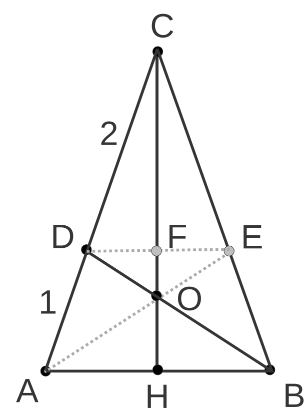
   Solution. Load vertices A, B and C with equal masses, m. Then, the center of mass (COM) of the three masses is at the intersection of the three medians, because it has to belong to each segment connecting the mass at the vertex of the triangle with the COM of the other two masses, i.e. the middle of the opposite side. COM this belongs to all three medians and is the centroid, O of the triangle. It divides each median in the 2:1 ratio because it is a COM of mass m at the vertex and a mass $2m$ at the middle of the opposite side.

2. **Problem.** In isosceles triangle ABC point D divides the side AC into segments such that $|AD|:|CD| = 1:2$. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio $|OH|$ to $|CH|$.



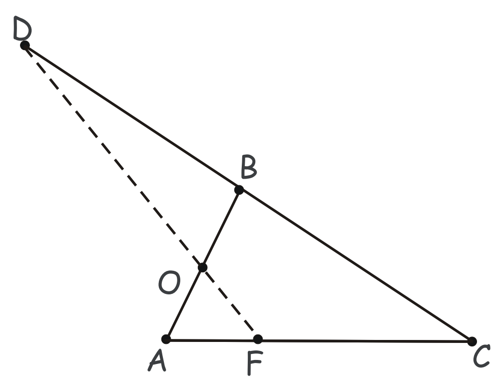
Solution.

a. Using the similarity and Thales theorem. First, let us perform a supplementary construction by drawing the segment DE parallel to AB, $DE||AB$, where point E belongs to the side CB, and point F to DE and the altitude CH. Notice the similar triangles, $AOH \sim DOF$, which implies, $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|}$. By Thales theorem, $\frac{|AH|}{|DF|} = \frac{|AC|}{|AD|} = 1 + \frac{|CD|}{|AD|} = \frac{3}{2}$, and $\frac{|OF|}{|OH|} = \frac{|DF|}{|AH|} = \frac{2}{3}$, so that $\frac{|FH|}{|OH|} = \frac{|FO|+|OH|}{|OH|} = \frac{5}{3}$. $\frac{|CH|}{|OH|} = \frac{|CH|}{|FH|}\frac{|FH|}{|OH|} = 3 \cdot \frac{5}{3} = 5$, because $\frac{|CH|}{|FH|} = 1 + \frac{|CF|}{|FH|} = 1 + \frac{|CD|}{|DA|}$. Therefore, the sought ratio is, $\frac{|OH|}{|CH|} = \frac{1}{5}$.



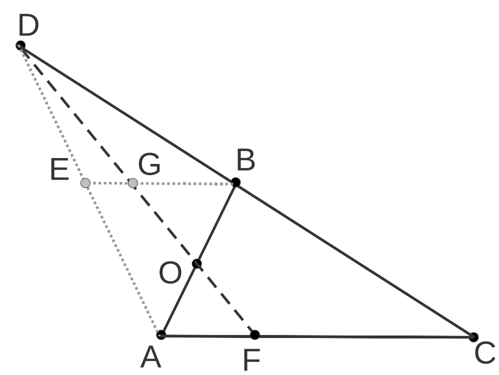
b. Using the Method of the Center of Mass. Load vertices A, B and C with masses $2m$, $2m$, and m, respectively. Then, H is the COM of masses at A and B, and D is the COM of masses at A and C, and O is the COM of all 3 masses in the vertices of the triangle ABC. Therefore, $|OC|:|OH| = (2m + 2m):m = 4:1$, $|OH|:|CH| = 1:5$.

3. **Problem.** Point D belongs to the continuation of side CB of the triangle ABC such that $|BD| = |BC|$. Point F belongs to side AC, and $|FC| = 3|AF|$. Segment DF intercepts side AB at point O. Find the ratio $|AO|:|OB|$.



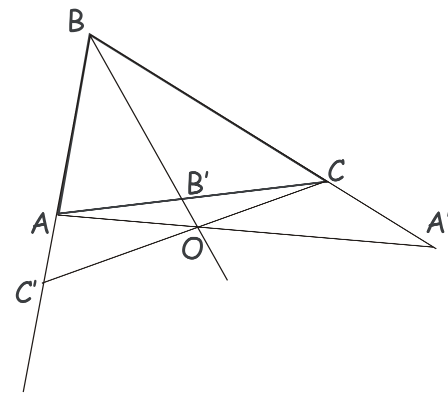
Solution.

a. Using the similarity and Thales theorem. First, let us perform a supplementary construction by drawing the segment BE parallel to AC, $BE||AC$, where E belongs to the side AD of the triangle ACD. BE is the mid-line of the triangle ACD, and, by Thales, also of AFD and FDC. Therefore, $|EG| = \frac{1}{2}|AF|$, $|GB| = \frac{1}{2}|FC|$ and $|EB| = \frac{1}{2}|AC|$, so $\frac{|BG|}{|EG|} = \frac{|FC|}{|AF|} = 3$. On the other hand, again, by Thales, or, noting similar triangles $AOF \sim BOG$, $\frac{|AO|}{|OB|} = \frac{|AF|}{|GB|} = 2\frac{|AF|}{|AC|} = \frac{2}{3}$.



b. Using the Method of the Center of Mass. Load vertices A, C and D with masses $3m$, m and m, respectively. Then, F is the center of mass (COM) of A and C, B is the COM of D and C, and O is the COM of the triangle ACD, $|AO|:|OB| = (m+m):3m = 2:3$.

Theorem (Extended Ceva). Segments (Cevians) connecting vertices A, B and C, with points A', B' and C' on the sides, or on the lines that suitably extend the sides BC, AC, and AB, of triangle ABC, are concurrent if and only if,



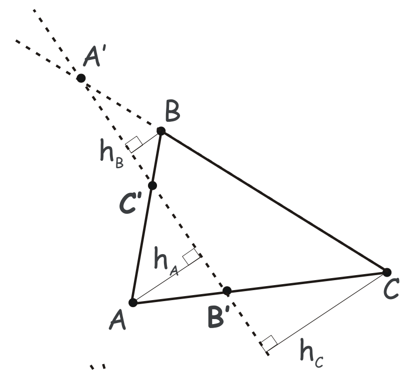
$$\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|}=1$$

Proof. We have already proven this theorem for the case when points A', B' and C' lie on the sides, but not on the lines extending the sides as it is shown in the figure. Let us now consider this latter case. Let us first load points A', B and C' with masses $m_{A'}$, m_B and $m_{C'}$ such that point A is the center of mass for m_B and $m_{C'}$, $m_B|AC'| = m_{C'}|AB|$, and point C is the COM for $m_{A'}$ and m_B, $m_{A'}|BC| = m_B|A'C|$. Then, the COM of all three masses at the vertices of the triangle $A'BC'$ is at the point O, which is the intersection of AA' and CC'. Let BO cross side AC at point B'. Adding mass to vertex B would move the COM of the three masses along line BO, because the COM of the initial 3 masses is at O. Let us add another mass m_B to vertex B, so that the total mass at this vertex is $2m_B$.The resulting system of masses then has the same COM as two masses, $m_B + m_{A'}$ and $m_B + m_{C'}$ at points A and C, respectively. This COM is common to AC and BO, and therefore is at point B', so $(m_B + m_{A'})|AB\,'| = (m_B + m_{C'})|B'C|$. Hence, we obtain,

$$\frac{|AC'|}{|C'B|}\frac{|BA'|}{|A'C|}\frac{|CB'|}{|B'A|}=\frac{1}{1+\frac{m_{C'}}{m_B}}\left(1+\frac{m_{A'}}{m_B}\right)\frac{m_B+m_{C'}}{m_B+m_{A'}}=1$$

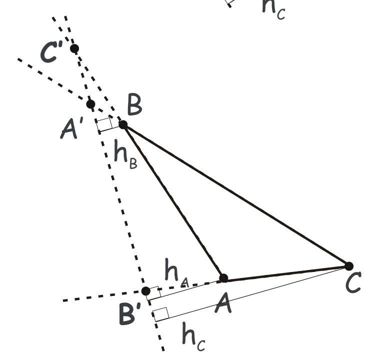
Theorem (Menelaus). Points A', B' and C' on the sides, or on the lines that suitably extend the sides BC, AC, and AB, of triangle ABC, are collinear (belong to the same line) if and only if,

$$\frac{|A'B|}{|A'C|}\frac{|B'C|}{|B'A|}\frac{|C'A|}{|C'B|} = 1$$



Menelaus's theorem provides a criterion for collinearity, just as Ceva's theorem provides a criterion for concurrence.

Proof (similarity). The statement could be proven with, or without using the method of point masses.



First, assume the points are collinear and consider rectangular triangles obtained by drawing perpendiculars onto the line A'B'. Using their similarity, one has

$$\frac{|A'B|}{|A'C|} = \frac{h_B}{h_C}, \frac{|B'C|}{|B'A|} = \frac{h_C}{h_A}, \frac{|C'A|}{|C'B|} = \frac{h_A}{h_B}$$

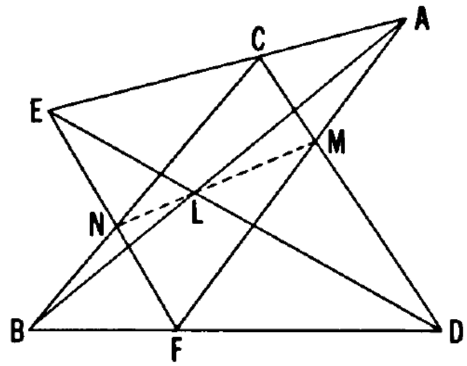
Wherefrom the statement of the theorem is obtained by multiplication (Coxeter & Greitzer).

Proof (point masses). Alternatively, let us load points A, A' and C in the upper Figure with the point masses m_1, m_2 and m_3, respectively. We select m_1, m_2 and m_3 such that B' is the COM of $m_1(A)$ and $m_3(C)$, and B is the COM of $m_2(A')$ and $m_3(C)$. The COM of all 3 masses belongs to both segments AB and $A'B'$, which means that it is at point C'. Then,

$$\frac{|A'B|}{|A'C|} = \frac{m_3}{m_2 + m_3}, \frac{|B'C|}{|B'A|} = \frac{m_1}{m_3}, \frac{|C'A|}{|C'B|} = \frac{m_2 + m_3}{m_1}$$

Wherefrom the Menelaus theorem is obtained by multiplication. The case shown in the lower figure is considered in a similar way.

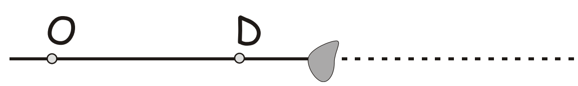
Theorem (Pappus). If A, C, E are three points on one line, B, D and F on another, and if three lines, AB, CD, EF, meet DE, FA, BC, respectively, then the three points of intersection, L, M, N, are collinear.



This is one of the most important theorems in planimetry, and plays important role in the foundations of projective geometry. There are a number of ways to prove it. For example, one can consider five triads of points, LDE, AMF, BCN, ACE and BDF, and apply Menelaus theorem to each triad. Then, appropriately dividing all 5 thus obtained equations, we can obtain the equation proving that LMN are collinear, too, also by the Menelaus theorem. However, one can prove the Pappus theorem directly, using the method of point masses.

Instead of simply proving the theorem, consider the following problem.

Problem. Using only pencil and straightedge, continue the line to the right of the drop of ink on the paper without touching the drop.



Solution by the Method of the Center of Mass.

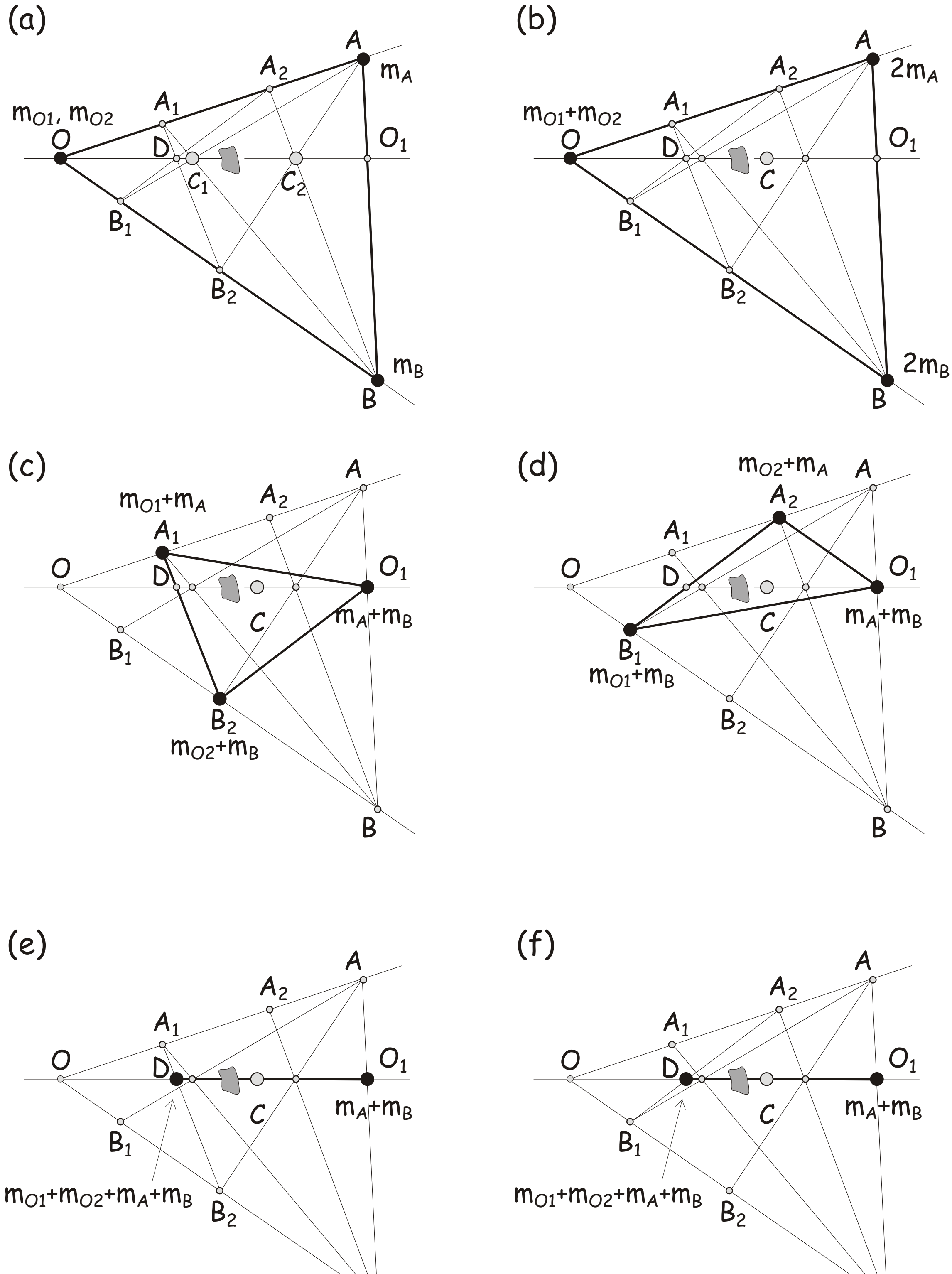
Construct a triangle OAB, which encloses the drop, and with the vertex O on the given line (OD). Let O_1 be the crossing point of (OD) and the side AB. Let us now load vertices A and B of the triangle with point masses m_A and m_B, such that their center of mass (COM) is at the point O_1. Then, each point of the (Cevian) segment OO_1 is the center of mass of the triangle OAB for some point mass m_O loaded on the vertex O. The (Cevian) segments from vertices A and B, which pass through the center of mass of the triangle C, connect each of these vertices with the center of mass of the other two vertices on the opposite side of the triangle, OB and OA, respectively.

For the mass m_{O1} loaded on the vertex O, the center of mass of the triangle is C_1, and the centers of mass of the sides OA and OB are A_1 and B_1, respectively.

Similarly, C_2, A_2 and B_2 are those for the mass m_{O2} on the vertex O. The center of mass of the side AB is always at the point O_1, independent of mass m_O.

If we can show that segments A_1B_2 and A_2B_1 cross the given line (OD) at the same point, D, then our problem is solved, as we can draw Cevians BA_2 and AB_2, whose crossing points are on the segment OO_1 on the other side of the drop, by sequentially drawing Cevians BA_1 and AB_1 and segments A_1B_2, B_1A_2, Figure 1(a).

Let us load vertices O, A and B with masses $m_{O1}+ m_{O2}$, $2m_A$ and $2m_B$, respectively, Figure 1(b). The center of mass of OAB is now at some point C, in-between C_1 and C_2 (actually, it is not important where it is on the line OO_1). Let us now move point masses m_{O1} and m_A to their center of mass A_1 on the side OA, m_{O2} and m_B to their center of mass B_2 on the side OB, and m_A and m_B to their center of mass O_1 on the side AB. Now masses are at the vertices of the triangle $A_1B_2O_1$ with the same center of mass, C, Figure 1(c). Consequently, the crossing point D of segments A_1B_2 and OO_1 is the center of mass for masses $m_{O1}+m_A$ and $m_{O2}+m_B$ placed at points A_1 and B_2, respectively. Point C then is the center of mass for $m_{O1}+m_{O2}+m_A +m_B$ at point D and $m_A +m_B$ at point O_1, Figure 1(e). Repeating similar arguments for the triangle $A_2B_1O_1$, Figure 1(d,f), we see that point D is also the crossing point of segments A_1B_2 and OO_1. Therefore, D is the crossing point of all three segments, A_1B_2, A_2B_1 and OO_1, which completes the proof.



(a)
A
A_2
m_A
m_{O1}, m_{O2}
A_1
O
D
O_1
C_1
C_2
B_1
B_2
m_B
B
(b)
A
A_2
$2m_A$
$m_{O1}+m_{O2}$
A_1
O
D
O_1
C
B_1
B_2
$2m_B$
B
(c)
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$m_{O1}+m_A$
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$m_{O1}+m_{O2}+m_A+m_B$
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B