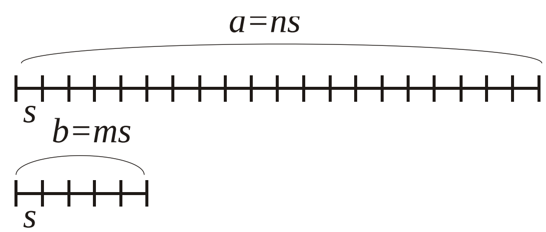
September 24, 2017

Geometry.

Commensurate and incommensurate segments. The Euclidean algorithm.

Definition. Two segments, a and b, are **commensurate** if there exists a third segment, s, such that it is contained in each of the first two segments a whole numbers of times with no remainder.



$$\{a, b\ are\ commensurate\} \Leftrightarrow \{\exists s, \{n, m\} \in N: (a = ns) \wedge (b = ms)\}$$

The segment s is called a common measure of the segments a and b. The concept of commensurability is similar to that of the common divisor for integers. It can be extended to any two quantities of the same denomination – two angles, two arcs of the same radius, or two weights.

The greatest common measure.

If a common measure s of two segments a and b is sub-divided into two, three, or, generally, any number of equal smaller segments, these smaller segments are also common measures of the segments a and b. In this way, an infinite set of common measures, decreasing in length, can be obtained, $\left\{\frac{s}{p}, p \in N\right\}$. Since any common measure is less than the smaller segment, $s \leq b$, there must be the largest among the common measures, which is called the greatest common measure.

Finding the greatest common measure (**GCM**) is done by the method of consecutive exhaustion called **Euclidean algorithm.** It is similar to the method of consecutive division used for finding the greatest common divisor in arithmetic. The method is based on the following theorem.

Theorem. Two segments a and b are commensurate, if and only if the smaller segment, b, is contained in the greater one a whole number of times with no remainder, or with a remainder, $r < b$, which is commensurate with the smaller segment, b.

$$\exists n \in N: (a = nb + r) \wedge \Big((r = 0) \vee \big(\exists s, \{p, q\} \in N: (b = ps) \wedge (r = qs)\big)\Big).$$

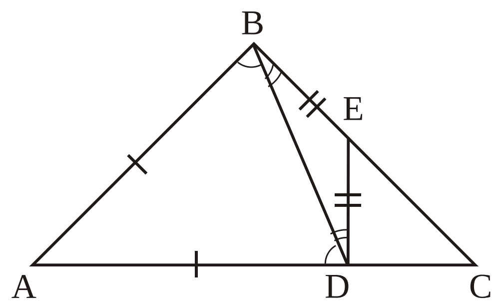
The greatest common measure of two segments is also the greatest common measure of the smaller segment and the remainder, or there is no remainder.

Proof. First, consider the necessary condition. Let a and b be commensurate, $\{\exists s, \{n, m\} \in N: (a = ns) \wedge (b = ms)\}$, and $a > b$. Let s be their greatest common measure. Then, either $s = b$ $(m = 1)$ and segment b is contained in a a whole number of times with no remainder, being the GCM of the two segments, or, $\exists k \in N: a = kb + r, 0 < r < b$. Then, $a = ns = kb + (n - km)s$, where $m < km < n$, and, therefore, $r = qs, \{q = n - km\} \in N$, which shows that r and b are commensurate. The sufficiency follows from the observations that (i) if segment b is contained in a a whole number of times with no remainder, then the segments are commensurate, and b is the greatest common measure of the two, while (ii) if $a = kb + r$, and b and r are commensurate with the greatest common measure s, $\exists\{p, q\} \in N: (b = ps) \wedge (r = qs)$, then $a = (kp + q)s = ns, n = kp + q \in N$, and a and b are also commensurate with the same GCM.

The Euclidean algorithm.

In order to find the GCM of the two segments, a and b, we can proceed as follows. First, using a compass exhaust the greater segment, marking on it the smaller segment as many times as possible, until the remainder is smaller than the smaller segment, b, or there is no remainder. According to Archimedes' exhaustion axiom, these are the only two possible outcomes. Following the above theorem, the problem now reduces to finding the GCM of this remainder, r_1, and the smaller segment, b. We now repeat the same procedure, exhausting segment b with r_1, and again, there is either no remainder and r_1 is the GCM of a and b, or there is a remainder $r_2 < r_1$. The problem is then reduced to finding the GCM of a pair of even smaller segments, r_1 and r_2, and so on. If segments a and b are commensurate and their GCM, s, exists, then this process will end after some number of steps, namely, on step n where $r_n = s$. Indeed, all remainders in this process are multiples of s, $\forall m: r_m = p_m s, p_m \in N$ and $p_1 > p_2 > \cdots > p_m > \cdots$ is the decreasing sequence of natural numbers, which necessarily terminates, since any non-empty set of positive integers has the smallest number ("principle of the smallest integer"). If the procedure never terminates, then segments a and b have no common measure and are incommensurate.

Example. The hypotenuse of an isosceles right triangle is incommensurate to its leg. Or, equivalently, the diagonal of a square is incommensurate to its side.

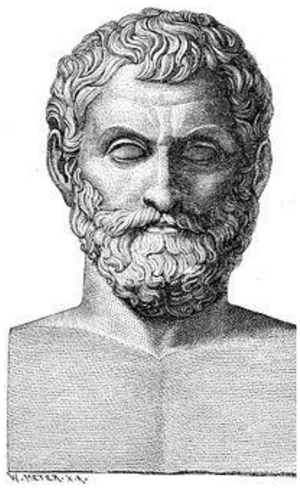


Proof. Consider the isosceles right triangle ABC shown in the Figure. Because the hypotenuse is less than twice the leg by the triangle inequality, the leg can only fit once in the hypotenuse, this is marked by the segment AD. Let the perpendicular to the hypotenuse at point D intercept leg BC at point E. Triangle BDE is also isosceles. This is because angles BDE and DBE supplement equal angles ADB and ABD to 90 degrees, and therefore are also equal. Triangle CDE is an isosceles right triangle, similar to ABC. Its leg |DC|=|AC|-|AB|=|DE|=|BE| is a remainder of subtracting the leg |AB|=|AD| from the hypotenuse, |AC|, while the hypotenuse, |CE| =|BC|-|BE|=|BC|-|DC|, is the remainder of subtracting this remainder from the leg |AB|=|BC|. Hence, on the second step of the Euclidean algorithm we arrive at the same problem as the initial one, only scaled down by some overall factor. Obviously, this process never ends, and therefore the hypotenuse |AC| and the leg |AB| are incommensurate.

Thales (intercept) theorem. Similarity and related concepts.

Megiston topos: hapanta gar chorei (Μέγιστον τόπος· άπαντα γαρ χωρεί)

”Space is the greatest thing, as it contains all things”



Thales of Miletus

Born c. 624 BC

Died c. 546 BC

Era Pre-Socratic

Thales of Miletus (/ˈθeɪliːz/; Greek: Θαλῆς (ὁ Μιλήσιος), Thalēs; c. 624 – c. 546 BC) was a pre-Socratic Greek philosopher from Miletus in Asia Minor, and one of the Seven Sages of Greece. Many, most notably Aristotle, regards him as the first philosopher in the Greek tradition.

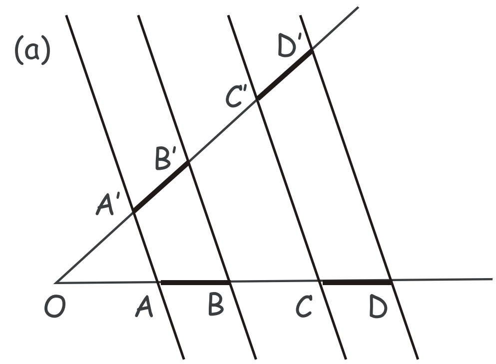
Thales was probably the first to introduce the **scientific method** into public discourse. He attempted to explain natural phenomena without reference to mythology and was tremendously influential in this respect. Thales' rejection of mythological explanations became an essential idea for the scientific revolution. He was also the first to **define general principles and set forth hypotheses**, and as a result has been dubbed the "Father of Science". Aristotle reported Thales' hypothesis about the nature of matter – that the originating principle of nature was a single material substance: water, the first **materialist** philosophy.

In mathematics, Thales is known for his contribution to geometry, both theoretical as well as practical. Thales understood similar triangles and right triangles, and used that knowledge in practical ways to solve problems such as calculating the height of pyramids and the distance of ships from the shore. The story is told that he measured the height of the pyramids by their shadows at the moment when his own shadow was equal to his height. He is also credited with the first use of deductive reasoning applied to geometry, by deriving four corollaries to Thales' Theorem. As a result, he has been hailed as the first true mathematician.

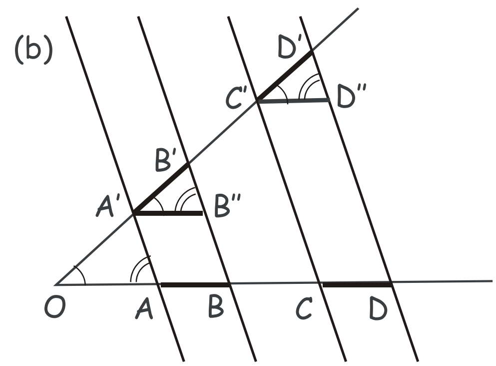
Thales (intercept) theorem.

Thales' intercept theorem (not to be confused with another theorem with that name, which is a particular case of the inscribed angle theorem) is an important theorem in elementary geometry about the ratios of various line segments that are created if two intersecting lines are intercepted by a pair of parallels. It is equivalent to the theorem about ratios in similar triangles.

Theorem 1. Let parallel lines AA', BB', CC' and DD' intercept the sides of an angle AOA' such that segments AB and CD on one side of the angle are congruent, |AB| = |CD|. Then the corresponding segments formed at the intersection of these lines with the other side of the angle are also congruent, |A'B'|=|C'D'|, Fig. 1(a).



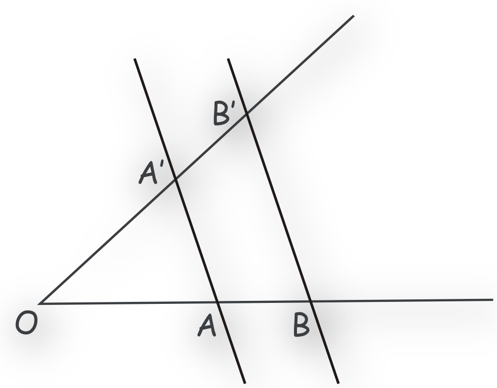
Proof. Draw lines A'B'' and C'D'' parallel to the side OA, such that AA'B''B and CC'D''D are parallelograms, Fig. 1(b). By the property of a parallelogram, |AB|=|A'B''|, and |CD|=|C'D''|. Angles B'A'B'' and D'C'D'' and A'B''B' and C'D''D' are formed by the parallel lines and therefore are congruent. Hence, triangles A'B''B' and C'D''D' are congruent, and therefore |A'B'|=|C'D'|.



Theorem 2. Let the sides of an angle AOA' be intercepted by two parallel lines AA' and BB', Fig. 2. Then, for the segments obtained by these intersections, the following holds.

1. The ratios of any 2 segments on the first line, OA, equal the ratios of the corresponding segments on the second line, OA',

$$\frac{|OA|}{|AB|} = \frac{|OA'|}{|A'B'|} \wedge \frac{|OB|}{|OA|} = \frac{|OB'|}{|OA'|} \wedge \frac{|OB|}{|AB|} = \frac{|OB'|}{|A'B'|}.$$

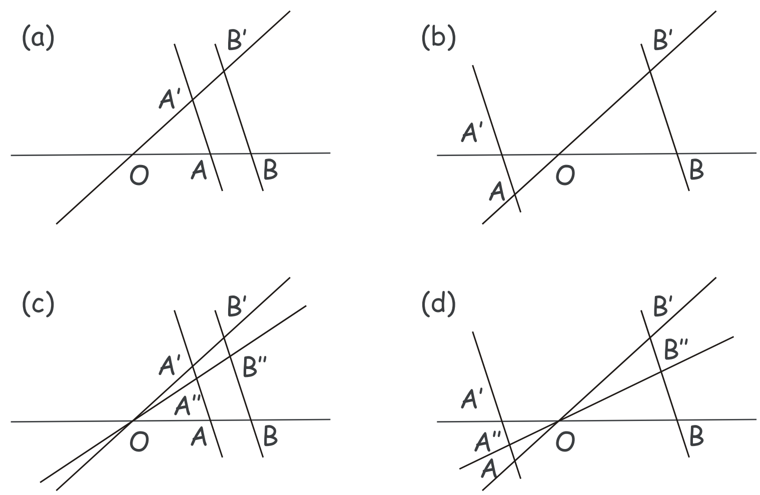


2. The ratio of the 2 segments on the same line

starting at O equals the ratio of the segments on the parallels,

$$\frac{|OA|}{|OB|} = \frac{|OA'|}{|OB'|} = \frac{|AA'|}{|BB'|}.$$

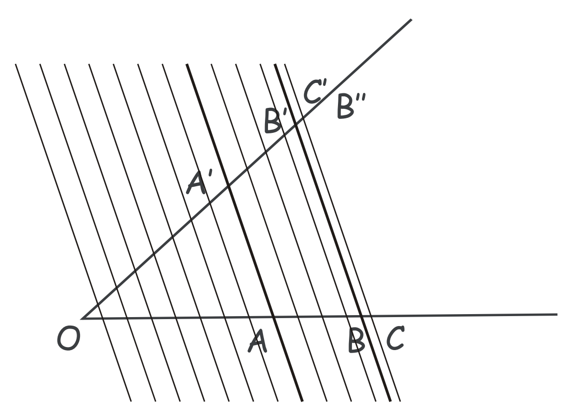
3. The converse of the first statement is true as well, i.e. if the 2 intersecting lines forming the sides of an angle with the vertex O are intercepted by 2 arbitrary lines at points A, B on one side and A', B' on the other, such that $\frac{|OA|}{|OB|} = \frac{|OA'|}{|OB'|}$ holds, then the 2 intercepting lines are parallel. However, the converse of the second statement is not true.
4. If you have more than 2 lines intersecting in O, then ratio of the 2 segments on a parallel equals the ratio of the according segments on the other parallel. Several examples of parallel lines configurations are shown in the Figure.



$$\frac{|AA''|}{|BB''|} = \frac{|A''A'|}{|B''B'|}, \frac{|AA''|}{|AA'|} = \frac{|BB''|}{|BB'|}.$$

Proof. We shall prove the statement 1 above, the rest follows straightforwardly.

According to Thales Theorem (Theorem 1 above), the intercept points of a set of parallel lines passing through the endpoints of an equal length segments on one side of an angle form a set of equal-length segments on the other side of the same angle. Consider the situation where parallel lines AA' and BB' intercept angle AOA', and assume that $\frac{|AB|}{|OA|} = \frac{|A'B'|}{|OA'|}$ does not hold. For definitiveness, let us assume that $\frac{|AB|}{|OA|} > \frac{|A'B'|}{|OA'|}$. Then, there exists point B'' belonging to the side



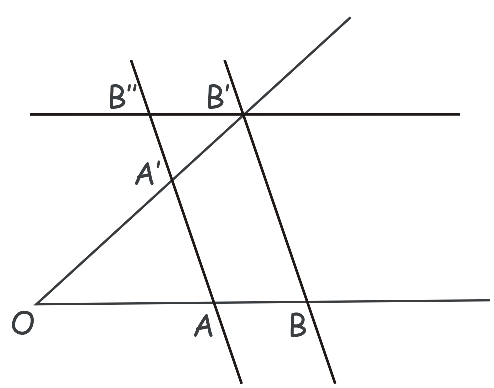
OA', such that |OB''|>|OB|, and $\frac{|AB|}{|OA|} = \frac{|A'B''|}{|OA'|}$.

Let us draw a set of lines parallel to AA' and BB', such that they divide segment OA' into a set of congruent segments of length $l < |B'B''|$. If we continue these lines past point A, there are two possibilities. Either segments OA' and OB' are commensurate and l is their common measure, then one of the lines must coincide with BB', or, the first such line passing farther from O than BB' is CC', and |OC'|<|OB''|.

In the first case, both OA and OB and AA' and BB' are divided into an equal number of congruent segments, and, therefore, $\frac{|AB|}{|OA|} = \frac{|A'B'|}{|OA'|}$ holds. In the second case, because all segments obtained at the intercepts of these lines with the sides of the angle are respectively congruent, $\frac{|AC|}{|OA|} = \frac{|A'C'|}{|OA'|}$. On the other hand, by construction we have |AC|>|AB| and |A'C'|<|A'B''|, so $\frac{|A'B''|}{|OA'|} > \frac{|A'C'|}{|OA'|} = \frac{|AC|}{|OA|} > \frac{|AB|}{|OA|}$, which contradicts our assumption. Another way to note a contradiction with our assumptions is $\frac{|AB|}{|OA|} < \frac{|AC|}{|OA|} = \frac{|A'C'|}{|OA'|} < \frac{|A'B''|}{|OA'|} = \frac{|AB|}{|OA|}$. Hence, $\frac{|AB|}{|OA|} = \frac{|A'B'|}{|OA'|}$ must hold.

Consequently, $\frac{|OB|}{|OA|} = 1 + \frac{|AB|}{|OA|} = 1 + \frac{|A'B'|}{|OA'|} = \frac{|OB'|}{|OA'|}$ also holds.

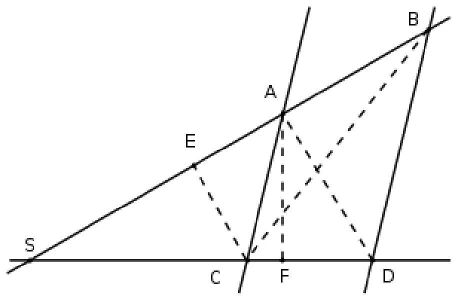
Exercise. Prove claim 2 of the theorem, i. e. $\frac{|OA|}{|OB|} = \frac{|OA'|}{|OB'|} = \frac{|AA'|}{|BB'|}$.



Hint: draw line B'B'' parallel to OB and apply the claim 1 proven above to the obtained segments on the angle OA'A.

Heuristic Alternate Proof of Thales Theorem

claim 1



Due to heights of equal length ($CA \parallel BD$) we have $|\triangle CDA| = |\triangle CBA|$ and therefore $|\triangle SCB| = |\triangle SDA|$. This yields

$$\frac{|\triangle SCA|}{|\triangle CDA|} = \frac{|\triangle SCA|}{|\triangle CBA|} \text{ and } \frac{|\triangle SCA|}{|\triangle SDA|} = \frac{|\triangle SCA|}{|\triangle SCB|}$$

Plugging in the actual formula for triangle areas ($\frac{base \cdot height}{2}$) transforms that into $\frac{|SC|}{|CD|} \cdot \frac{|AF|}{|AF|} = \frac{|SA|}{|AB|} \cdot \frac{|EC|}{|EC|} \wedge \frac{|SC|}{|SD|} \cdot \frac{|AF|}{|AF|} = \frac{|SA|}{|SB|} \cdot \frac{|EC|}{|EC|}$

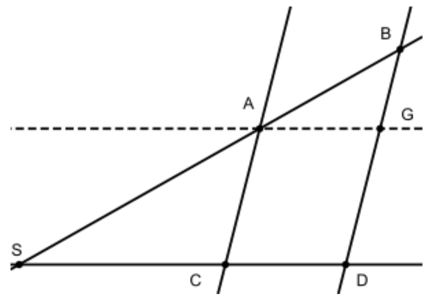
Cancelling the common factors results in:

(a) $\frac{|SC|}{|CD|} = \frac{|SA|}{|AB|}$ and (b) $\frac{|SC|}{|SD|} = \frac{|SA|}{|SB|}$.

Now use (b) to replace $|SA|$ and $|SC|$ in (a): $\frac{\frac{|SA| \cdot |SD|}{|SB|}}{|CD|} = \frac{\frac{|SB| \cdot |SC|}{|SD|}}{|AB|}$

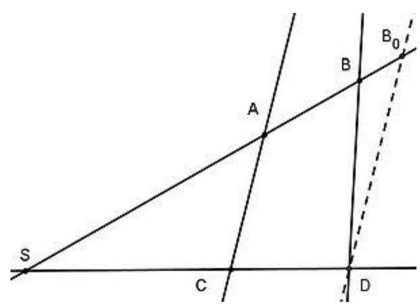
Using (b) again this simplifies to: (c) $\frac{|SD|}{|CD|} = \frac{|SB|}{|AB|}$ □

claim 2



Draw an additional parallel to SD through A. This parallel intersects BD in G. Then you have $|AC| = |DG|$ and due to claim 1 $\frac{|SA|}{|SB|} = \frac{|DG|}{|BD|}$ and therefore $\frac{|SA|}{|SB|} = \frac{|AC|}{|BD|}$.

claim 3



Assume AC and BD are not parallel. Then the parallel line to AC through D intersects SA in $B_0 \neq B$. Since $|SB| : |SA| = |SD| : |SC|$ is true, we have

$$|SB| = \frac{|SD| \cdot |SA|}{|SC|}$$

and on the other hand from claim 2 we have

$|SB_0| = \frac{|SD|\cdot|SA|}{|SC|}$. So B and B_0 are on the same side of S and have the same distance to S, which means $B = B_0$. This is a contradiction, so the assumption could not have been true, which means AC and BD are indeed parallel □

claim 4

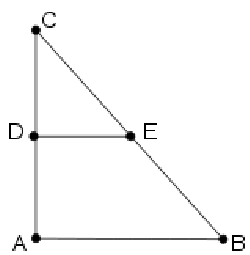
Can be shown by applying the intercept theorem for 2 lines.

Related Concepts

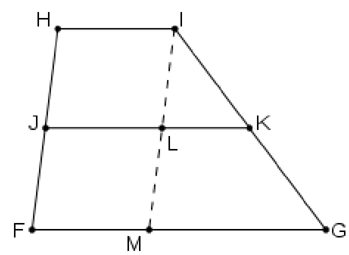
Parallel Lines in Triangles and Trapezoids

The intercept theorem can be used to prove that a certain construction yields a parallel line (segment).

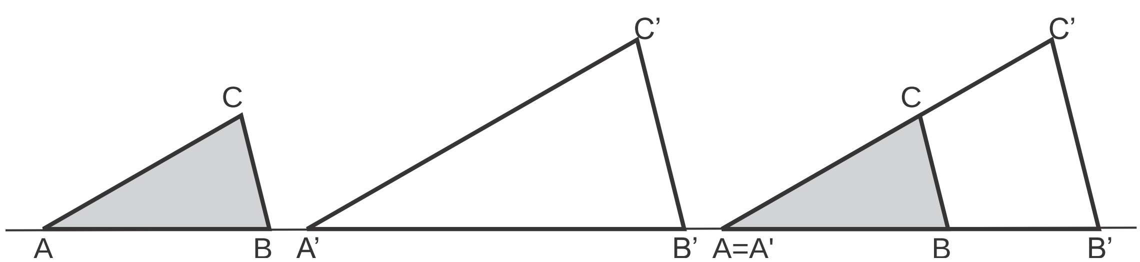
If the midpoints of 2 triangle sides are connected then the resulting line segment is parallel to the 3rd triangle side.



If the midpoints of 2 the non parallel sides of a trapezoid are connected, then the resulting line segment is parallel to the other 2 sides of the trapezoid.



Similarity and similar Triangles



Arranging 2 similar triangles, so that the intercept theorem can be applied

The intercept theorem is closely related to similarity. In fact it is equivalent to the concept of similar triangles, i.e. it can be used to prove the properties of similar triangles and similar triangles can be used to prove the intercept theorem. By matching identical angles you can always place 2 similar triangles in one another, so that you get the configuration in which the intercepts applies and vice versa the intercept theorem configuration contains always 2 similar triangles.

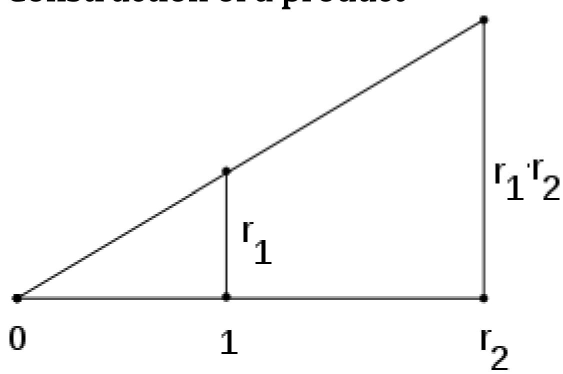
Algebraic formulation of Compass and Ruler Constructions

There are 3 famous problems in elementary geometry, which were posed by the Greek in terms of Compass and straightedge constructions.

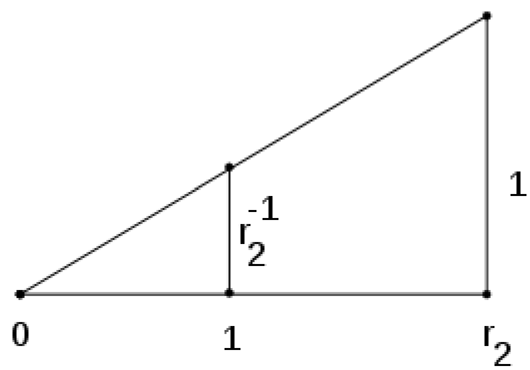
1. Trisecting the angle
2. Doubling the cube
3. Squaring the circle

Their solution took more than 2000 years until all 3 of them finally were settled in the 19th century using algebraic methods that had become available during that period of time. In order to reformulate them in algebraic terms using field extensions, one needs to match field operations with compass and straightedge constructions. In particular it is important to assure that for 2 given line segments, a new line segment can be constructed such that its length equals the product of lengths of the other two. Similarly one needs to be able to construct, for a line segment of length d, a new line segment of length d^{-1}. The intercept theorem can be used to show that in both cases the construction is possible.

Construction of a product

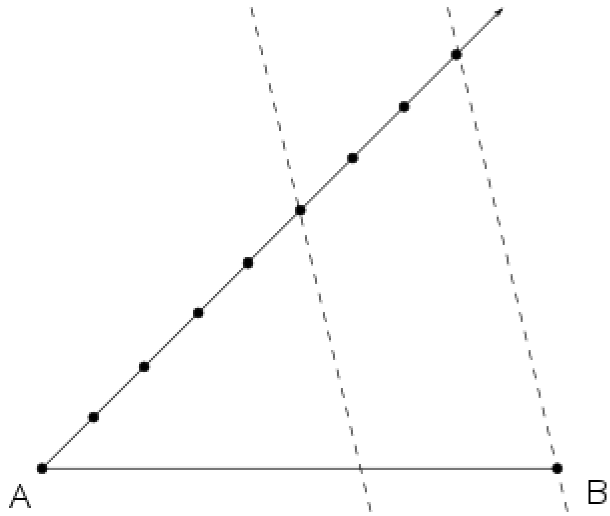


Construction of an Inverse



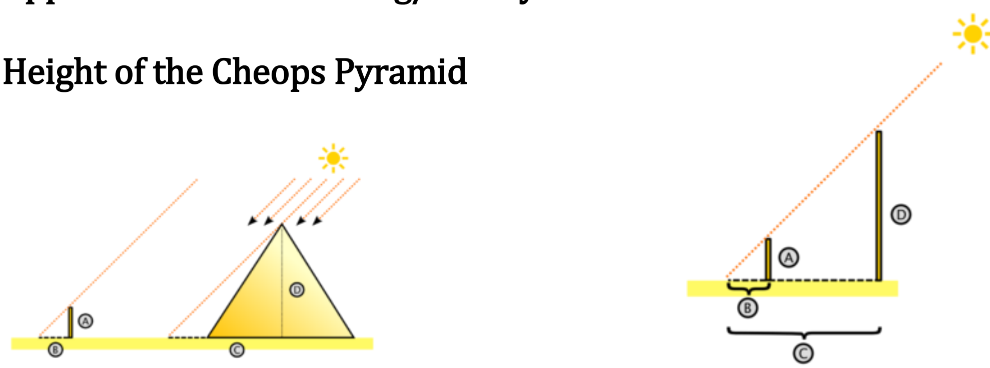
Dividing a line segment in a given ratio

To divide an arbitrary line segment $\overline{AB}$ in a $m:n$ ratio you draw an arbitrary angle in A with $\overline{AB}$ as one leg. One other leg you construct $m+n$ equidistant points, then you draw line through the last point and B and parallel line through the mth point. This parallel line divides $\overline{AB}$ in the desired ratio. The graphic to the right shows the partition of a line sgement $\overline{AB}$ in a 5:3 ratio.



Applications to Measuring/Survey

Height of the Cheops Pyramid



Figures illustrate measuring pieces and computing C and D

According to some historical sources the Greek mathematician Thales applied the intercept theorem to determine the height of the Cheops' pyramid. The following description illustrates the use of the intercept theorem to compute the height of the Cheops' pyramid, it does however not recount Thales' original work, which was lost.

He measured length of the pyramid's base and the height of his pole. Then at the same time of the day he measured the length pyramid's shadow and the length of the pole's shadow. This yields him the following data to work with:

- height of the pole (A): 1.63m
- shadow of the pole (B): 2m

- length of the pyramid base: 230m
- shadow of the pyramid: 65m

From this he computed

$$C = 65m + \frac{230m}{2} = 180m$$

Knowing A,B and C he was now able to apply the intercept theorem to compute

$$D = \frac{C \cdot A}{B} = \frac{1.63m \cdot 180m}{2m} = 146.7m$$

Measuring the Width of a River

The intercept theorem can be used determine a distance that cannot be measured directly, such as the width of a river or a lake, tall buildings or similar. The graphic to right illustrates the measuring of the width of a river. The segments $|CF|$, $|CA|$, $|FE|$ are measured and used to compute the wanted distance $|AB| = \frac{|AC||FE|}{|FC|}$.