MATH 8

ASSIGNMENT 17: AXIOMS OF GEOMETRY

FEB 11TH, 2018

This week, we are starting Geometry. Please buy: "The Art of Problem Solving, Volume 1: The Basics" by Sandor Lehoczky and Richard Rusczyk, ISBN-13: 978-0-9773045-6-1. Please do not get the solution manual (or hide it for emergencies). It is important that you try the problems on your own.

1. Euclid's Axioms

Near the beginning of the first book of the Elements, Euclid gives five postulates (axioms) for plane geometry, stated in terms of constructions (as translated by Thomas Heath);

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce [extend] a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance [radius].
4. That all right angles are equal to one another.
5. The parallel postulate: That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

2. Hilbert's Axioms

1. I. Incidence
   (a) For every two points A and B there exists a line a that contains them both. We write $AB = a$ or $BA = a$. Instead of "contains", we may also employ other forms of expression; for example, we may say "A lies upon a", "A is a point of a", "a goes through A and through B", "a joins A to B", etc. If A lies upon a and at the same time upon another line b, we make use also of the expression: "The lines a and b have the point A in common", etc.
   (b) For every two points there exists no more than one line that contains them both; consequently, if $AB = a$ and $AC = a$, where $B \neq C$, then also $BC = a$.
   (c) There exist at least two points on a line. There exist at least three points that do not lie on the same line.
   (d) For every three points A, B, C not situated on the same line there exists a plane α that contains all of them. For every plane there exists a point which lies on it. We write $ABC = \alpha$. We employ also the expressions: "A, B, C, lie in α"; "A, B, C are points of α", etc.
   (e) For every three points A, B, C which do not lie in the same line, there exists no more than one plane that contains them all.
   (f) If two points A, B of a line a lie in a plane α, then every point of a lies in α. In this case we say: "The line a lies in the plane α", etc. If two planes α, β have a point A in common, then they have at least a second point B in common.
   (g) There exist at least four points not lying in a plane.

2. Order
   (a) If a point B lies between points A and C, B is also between C and A, and there exists a line containing the distinct points A, B, C.
   (b) If A and C are two points, then there exists at least one point B on the line AC such that C lies between A and B.
   (c) Of any three points situated on a line, there is no more than one which lies between the other two.
   (d) Pasch's Axiom: Let A, B, C be three points not lying in the same line and let a be a line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.

3. Congruence

(a) If A, B are two points on a line a, and if A' is a point upon the same or another line a' , then, upon a given side of A' on the straight line a' , we can always find a point B' so that the segment AB is congruent to the segment A'B' . We indicate this relation by writing AB A' B'. Every segment is congruent to itself; that is, we always have $AB \cong AB$. We can state the above axiom briefly by saying that every segment can be laid off upon a given side of a given point of a given straight line in at least one way.

(b) If a segment AB is congruent to the segment $A'B'$ and also to the segment $A''B''$, then the segment $A'B'$ is congruent to the segment $A''B''$; that is, if $AB \cong A'B'$ and $AB \cong A''B''$, then $A'B' \cong A''B''$.

(c) Let AB and BC be two segments of a line a which have no points in common aside from the point B, and, furthermore, let A'B' and B'C' be two segments of the same or of another line a' having, likewise, no point other than B' in common. Then, if $AB \cong A'B'$ and $BC \cong B'C'$, we have $AC \cong A'C'$.

(d) Let an angle $\angle(h, k)$ be given in the plane α and let a line a' be given in a plane α'. Suppose also that, in the plane α', a definite side of the straight line a' be assigned. Denote by h' a ray of the straight line a' emanating from a point O of this line. Then in the plane α' there is one and only one ray k' such that the angle $\angle(h, k)$, or $\angle(k, h)$, is congruent to the angle $\angle(h', k')$ and at the same time all interior points of the angle $\angle(h', k')$ lie upon the given side of a'. We express this relation by means of the notation $\angle(h, k) \cong \angle(h', k')$.

(e) If the angle $\angle(h, k)$ is congruent to the angle $\angle(h', k')$ and to the angle $\angle(h'', k'')$, then the angle $\angle(h', k')$ is congruent to the angle $\angle(h'', k'')$; that is to say, if $\angle(h, k) \cong \angle(h', k')$ and $\angle(h, k) \cong \angle(h'', k'')$, then $\angle(h', k') \cong \angle(h'', k'')$.

(f) If, in the two triangles ABC and $A'B'C'$ the congruences $AB \cong A'B'$, $AC \cong A'C'$, $\angle BAC \cong \angle B'A'C'$ hold, then the congruence $\angle ABC \cong \angle A'B'C'$ holds (and, by a change of notation, it follows that $\angle ACB \cong \angle A'C'B'$ also holds).

4. Parallels (Euclid's Axiom): Let a be any line and A a point not on it. Then there is at most one line in the plane, determined by a and A, that passes through A and does not intersect a.

5. Continuity

(a) Axiom of Archimedes. If AB and CD are any segments then there exists a number n such that n segments CD constructed contiguously from A, along the ray from A through B, will pass beyond the point B.

(b) Axiom of line completeness. An extension of a set of points on a line with its order and congruence relations that would preserve the relations existing among the original elements as well as the fundamental properties of line order and congruence that follows from Axioms 1-3 and from 5a is impossible.