MATH 8
ASSIGNMENT 12: PRIME FACTORIZATIONS

JAN 7, 2017

Prime factorization

Theorem. *Any positive integer $m > 1$ can be written in the form*

$$m = p_1^{k_1} \dots p_l^{k_l}$$

where p_i are distinct prime numbers. Moreover, such a decomposition is unique (except for for reordering factors)

Problems

When doing this homework, be careful that you only use the material we had proved or discussed so far — in particular, please do not use the prime factorization. And I ask that you only use integer numbers — no fractions or real numbers.

1. Let $m = p_1^{a_1} \dots p_l^{a_l}$, $n = p_1^{b_1} \dots p_l^{b_l}$ (the prime factors are the same). Show that m is a divisor of n if and only if $a_1 \le b_1$, $a_2 \le b_2$, $\dots$, $a_l \le b_l$.

2. (a) List all divisors of $2^3 \cdot 7^2$.
(b) List all divisors of $2 \cdot 3^3 \cdot 5^2$.
(c) Let $m = p_1^{k_1} \dots p_l^{k_l}$. How many divisors does m have (including 1 and itself)?

3. (a) What is the GCD and LCM of $2^3 \cdot 3^2 \cdot 5$ and $2 \cdot 5^2 \cdot 7$?
(b) Let $m = p_1^{a_1} \dots p_l^{a_l}$, $n = p_1^{b_1} \dots p_l^{b_l}$ (the prime factors are the same). What is the GCD of m, n? what is the LCM?

4. Let us call a function $f(m)$ *multiplicative* if $f(mn) = f(m)f(n)$ for any pair of relatively prime numbers m, n.

Prove that the following functions are multiplicative:
(a) $t(n)$ = number of divisors of n (including 1 and itself)
(b) $s(n)$ = sum of all divisors of n
Use it to compute $s(81000)$.

5. (Many of you know this one, but for those who have never seen it before...)
We put on the desk cards with numbers 1–100, in a row, face up.
On the first turn, we turn over all cards.
On the second, we turn over every other card (2, 4, ...)
On the third turn, we turn over every card with a number which is multiple of 3.
Which cards will be face up at the end?

6. Anya goes to the rink once every two weeks. Betsy goes once every 12 days. Carolyne comes to the rink once in 3 weeks.

On Jan 1, all three of them came to the rink.
(a) How many time in this year will Betsy and Anya meet on the rink?
(b) How many times will all three of them meet on the rink this year?
(c) How many days will there be in the year when exactly two of them will be on the rink?