MATH 8
ASSIGNMENT 7: EUCLID'S ALGORITHM CONTINUED
NOV 5TH, 2017

Today we continued discussion of Euclid's algorithm for computing the greatest common divisor of two numbers (a, b).

1. If needed, switch the two numbers so that $a > b$
2. Compute the remainder r upon division of a by b. Replace pair (a, b) with the pair (b, r)
3. Repeat the previous step until you get a pair of the form $(d, 0)$. Then $\gcd(a, b) = \gcd(d, 0) = d$.

This algorithm also gives the following result:

Theorem. 1. *An integer m is a common divisor of a, b if and only if m is a divisor of $d = \gcd(a, b)$.*
2. *An integer m can be written in the form $m = ax + by$ if and only if m is the multiple of $\gcd(a, b)$.*

For example:

$$\begin{aligned}\gcd(42, 100) &= \gcd(42, 16) \qquad\qquad (\text{because } 100 = 2 \cdot 42 + 16)\\ &= \gcd(16, 10) = \gcd(10, 6) = \gcd(6, 4)\\ &= \gcd(4, 2) = \gcd(2, 0) = 2\end{aligned}$$

which gives:

$$\begin{aligned}16 &= 100 - 2 \cdot 42\\ 10 &= 42 - 2 \cdot 16 = 42 - 2(100 - 2 \cdot 42) = -2 \cdot 100 + 5 \cdot 42\\ 6 &= 16 - 10 = (100 - 2 \cdot 42) - (-2 \cdot 100 + 5 \cdot 42) = 3 \cdot 100 - 7 \cdot 42\\ 4 &= 10 - 6 = (-2 \cdot 100 + 5 \cdot 42) - (3 \cdot 100 - 7 \cdot 42) = -5 \cdot 100 + 12 \cdot 42\\ 2 &= 6 - 4 = (3 \cdot 100 - 7 \cdot 42) - (-5 \cdot 100 + 12 \cdot 42) = 8 \cdot 100 - 19 \cdot 42\end{aligned}$$

Thus, to write, say, 18 in the form $x \cdot 100 + y \cdot 42$, we notice that $18 = 9 \cdot 2$, so we can multiply both sides of equality $2 = 8 \cdot 100 - 19 \cdot 42$ by 9:

$$18 = 72 \cdot 100 - 171 \cdot 42$$

Problems

When doing this homework, be careful that you only used the material we had proved or discussed so far — in particular, please do not use the prime factorization. And I ask that you only use integer numbers — no fractions or real numbers.

1. Use Euclid's algorithm to find gcd of the following numbers. Also, write the gcd as a linear combination of a, b (except part (c)).
   (a) 7 and 30
   (b) 57 and 93
   (c) 1028 and 213
2. For each of the following equations, find at least one solution (in integer numbers) it it exists. If not, explain why it doesn't exist
$$\begin{aligned}31x + 27y &= 1\\ 58x + 38y &= 2\\ 58x + 38y &= 6\\ 58x + 38y &= 3\end{aligned}$$
3. You have two cups, one 240 ml, the other 180 ml. What amounts of water can be measured using these two cups?
4. Find at least one integer number x such that $12x$ gives remainder 5 when divided by 19. [Hint: this is equivalent to solving $12x - 19y = 5$.]