# MATH 8
# ASSIGNMENT 6: EUCLID'S ALGORITHM
OCT 29TH, 2017

## Notation

$\mathbb{Z}$ — all integers
$\mathbb{N}$ — positive integers: $\mathbb{N} = \{1, 2, 3 \dots\}$.
$d|a$ means that $d$ is a divisor of $a$, i.e., $a = dk$ for some integer $k$.
$\gcd(a, b)$: greatest common divisor of $a, b$.

## Euclid's algorithm

In the last assignment, we proved the following:

**Theorem.** *If $a = bq + r$, then the common divisors of pair $(a, b)$ are the same as common divisors of pair $(b, r)$. In particular,*

$$\gcd(a, b) = \gcd(b, r)$$

This gives a very efficient way of computing the greatest common divisor of $(a, b)$, called Euclid's algorithm:

**1.** If needed, switch the two numbers so that $a > b$
**2.** Compute the remainder $r$ upon division of $a$ by $b$. Replace pair $(a, b)$ with the pair $(b, r)$
**3.** Repeat the previous step until you get a pair of the form $(d, 0)$. Then $\gcd(a, b) = \gcd(d, 0) = d$.

For example:

$$\begin{aligned} \gcd(42, 100) &= \gcd(42, 16) \qquad\qquad (\text{because } 100 = 2 \cdot 42 + 16) \\ &= \gcd(16, 10) = \gcd(10, 6) = \gcd(6, 4) \\ &= \gcd(4, 2) = \gcd(2, 0) = 2 \end{aligned}$$

## Problems

When doing this homework, be careful that you only used the material we had proved or discussed so far — in particular, please do not use the prime factorization. And I ask that you only use integer numbers — no fractions or real numbers.

**1.** Use Euclid's algorithm to compute $\gcd(54, 36)$; $\gcd(97, 83)$; $\gcd(1003, 991)$

**2.** Use Euclid's algorithm to find **all** common divisors of 2634 and 522.

**3.** Write each of the numbers appearing in the computation of $\gcd(100, 42)$ above in the form $k \cdot 100 + l \cdot 42$, for some integers $k, l$. For example,

$$\begin{aligned} 16 &= 1 \cdot 100 - 2 \cdot 42, \\ 10 &= 42 - 2 \cdot 16 = 42 - 2(100 - 2 \cdot 42) = \dots \end{aligned}$$

**4.** (a) Compute $\gcd(14, 8)$ **using Euclid's algorithm**
(b) Write $\gcd(14, 8)$ in the form $8k + 14l$. (You can use guess and check, or proceed in the same way as in the previous problem)
(c) Does the equation $8x + 14y = 18$ have integer solutions? Can you find at least one solution?
(d) Does the equation $8x + 14y = 17$ have integer solutions? Can you find at least one solution?
(e) Can you give complete answer, for which integer values of $c$ the equation $8x + 14y = c$ has integer solutions?

**5.** If I only have 15-cent coins and 12-cent coins, can I pay \$1.35? \$1.37?

**6.** (a) Show that any composite number has a prime divisor.
(b) Show that every number $n > 1$ can be written as a product of primes.

**7.** (a) Show that if $a$ is odd, then $\gcd(a, 2b) = \gcd(a, b)$.
*(b) Show that for $m, n \in \mathbb{N}$, $\gcd(2^n - 1, 2^m - 1) = 2^{\gcd(m,n)} - 1$