MATH 8
ASSIGNMENT 5: DIVISIBILITY

OCT 22, 2017

Divisibility

Notation:
$\mathbb{Z}$ — all integers
$\mathbb{N}$ — positive integers: $\mathbb{N} = \{1, 2, 3 \dots\}$.
We write $d|a$ if d is a divisor of a, i.e., $a = dk$ for some integer k. For example, $6|30$
We will frequently use (without proof) division with remainder:
for any integer a and positive integer n, we can find q, r such that

$$a = qn + r, \qquad 0 \le r < n \tag{1}$$

Moreover, q and r are uniquely determined: they are called quotient and remainder upon division of a by n.

Problems

1. Show that if $a|b$ and $b|c$, then $a|c$. For example: $6|30$, and $30|240$, so $6|240$.

2. Show that if a, b are divisible by d, then each of the following numbers is divisible by d:
(a) $a + b$
(b) $5a + 3b$
(c) any number of the form $na + mb$, with integer n, m.
(d) remainder r upon division of a by b

3. Let $a = qb + r$.
(a) Show that then each common divisor of a, b is also a divisor of r.
(b) Conversely, show that if d is a common divisor of b, r then it is also a divisor of a.

4. Show that if $p_1, \dots, p_k$ are prime, then the number $p_1 p_2 \dots p_k + 1$ is not divisible by any of p_i.
Deduce from this that there are infinitely many primes.

5. Show that if n is a positive integer, then $n^2 + 8n + 17$ is not divisible by $n + 4$.

6. (a) Show that for any integer n, $n^{2012} - 1$ is divisible by $n - 1$. [Hint: geometric progression!]
(b) Show that for any integer n, $n^{2013} + 1$ is divisible by $n + 1$. [Hint: write $n = -m$.]

7. Compute $(\sqrt{2} + \frac{1}{\sqrt{2}})^4$. Can you write it in the form $x + \sqrt{2}y$, with rational x, y?

8. Find the constant term of $(x + x^{-1})^{20}$. What about $(x + x^{-1})^{21}$?

***9.** What are the first 100 digits after the decimal point in the number $S = (\sqrt{26} + 5)^{100}$?
[Hint: $(\sqrt{26} - 5)^{100}$ is a really small number...]