CHAPTER 9

ISOMETRIES OF THE PLANE: SIMPLE, COMPOSITIONS

A transformation T is an operation which sends every point P of the plane to a new point, $P' = T(P)$. P is the "pre-image" of the "image" P'.

A transformation is an isometry if it preserves distances: for any points P, Q, we have T(P)T(Q)=PQ

Theorem *Reflections, rotations, and translations are isometries of the plane.*

1. *Any isometry sends lines to lines: if l is a line and T an isometry, then T(l) is again a line.*
2. *The composition of isometries is again an isometry.*

9.1 Translations - Isometries

Given a vector $\vec{v}$, we define the translation $T_{\vec{v}}$ to be the operation that adds to each point the vector $\vec{v}$: a point P is sent to a point $P' = T_{\vec{v}}(P) = (x_P, y_P) + \vec{v}$, i.e.$\overrightarrow{PP'} = \vec{v}$.

Theorem *A translation is an isometry, i.e. it preserves distances.*

Sketch of a proof: Take the simpler case of a translation along the x-axis. Take two points P,Q. Express the formula of the distance PQ, $d(P,Q) = x_Q - x_P$. Compare it to the distance $P'Q'$, $d(P',Q') = x_{Q'} - x_{P'} = x_Q + \vec{v} - (x_P + \vec{v}) = x_Q - x_P$. Generalize your proof using Pythagorean theorem, or distance formula for a general translation.

9.2 Dilations- Not Isometries

A dilation with center O and a scale factor of k is a transformation that maps every point P in the plane to point P' so that the ratio of the distance from the center of dilation to any point on the image compared to the distance from the center of dilation to the corresponding point on the pre-image will result in the scale factor, k. We define the translation of center O and scale factor k, $D_{O,k}$ to be the operation that multiplies each coordinate of a point P by a factor k, along the vector $\vec{v} = \overrightarrow{OP} = (x - x_O, y - y_O)$: a point P is sent to a point P' so that $P' = D_{O,k}(P) = \vec{v} + (kx, ky) = (x_O, y_O) + (k(x - x_O), k(y - y_O))$.

Theorem *A dilated line that that does not pass through the center of dilation results in a parallel line. A dilated line that passes through the center of dilation stays the same.*
Why? Dilation constructs similar triangles, leading to parallel lines.

Theorem *Triangle Proportionality (Side-Splitter) Theorem: In a triangle a line intersecting two sides is parallel to the third side iff it divides the first two sides proportionally.*

Derive a proof:

Algorithm to determine the center of dilation:

Lines drawn through each point on the pre-image and its corresponding image point will intersect at the center of dilation.

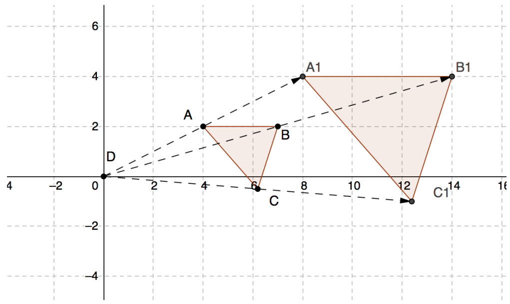


Figure 9.1: A dilation of factor $\alpha > 1$ about the origin

Theorem *Dilation is a similarity transformation, i.e. it preserves slopes. As a consequence angle measures, parallelism (things that were parallel are still parallel) and collinearity (points on a line, remain on the line)are preserved.*

Sketch of a proof: Take the simpler case of a dilation with the center in origin. Take two points P,Q. Express the formula of the slope of the line PQ, $m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P}$. *Compare it to the the slope of the line* $P'Q'$, $m_{P'Q'} = \frac{y_{Q'} - y_{P'}}{x_{Q'} - x_{P'}} = \frac{ky_Q - ky_P}{kx_Q - kx_P} = \frac{k(y_Q - y_P)}{k(x_Q - x_P)} = \frac{y_Q - y_P}{x_Q - x_P}$. *Generalize your proof using Pythagorean theorem, or distance formula for a general translation.*

Example: *The equation of a line l is $2x + y = 1$. The line l is the image of line m after a dilation of scale factor 3 with respect to the origin. What is the equation of the original line m?*

2x + y = 1, does not pass through the center of dilation, the origin. Its y-intercept is (0,1). The slope of the dilated line, m, stays the same, i.e. 2. The dilation center is the origin, thus all points on line h, including its y-intercept,(0,1), are dilated by a scale factor of 4. Therefore, the y-intercept of the dilated line is (0,4). Then, the resulting dilated line is: y = 2x + 4.

9.3 Rotations

A rotation has a single fixed point, i.e. the center of rotation around which everything else rotates. The amount of turning is called the rotation angle. The rotation center can be inside the figure and then the figure stays where it is and just spins. When the point is outside the figure, the figure moves along a circular arc (like an orbit) around the center of rotation.

Example: *Let us consider a "pre-image" triangle $\triangle ABC$ has been rotated to create image triangle $\triangle A'B'C'$. Find the center of rotation.*

Hint: How can one find the center of a circle given two chords?

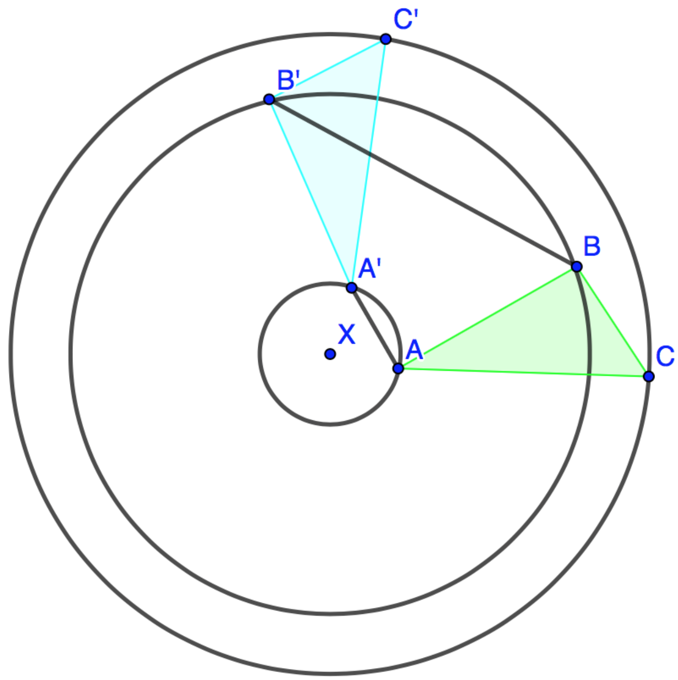
Hint: How can one find the center of two concentric circles given a chord in each of them?

Derive a proof:

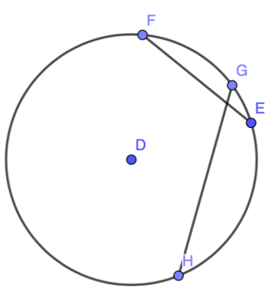
Review: The perpendicular bisector is the geometric place, "locus" of the points situated at an equal distance from the endpoints of a segment.

Algorithm:

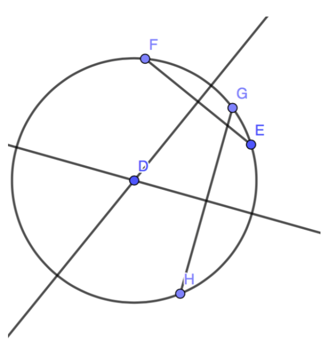
1. *Draw two segments, for example segment AA', BB'*
2. *Draw the perpendicular bisectors of the chords*
3. *The center of rotation is the intersection of the perpendicular bisectors.*



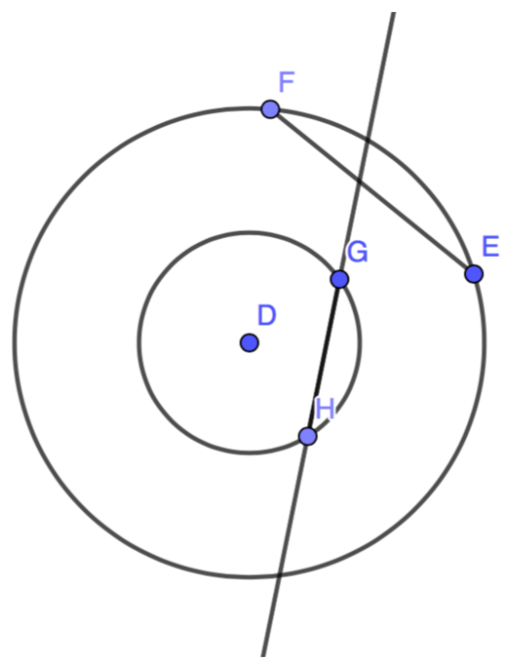
(a) Rotated Figure.



(b) Center given 2 chords



(c) Center given 2 chords



(d) Center given 2 chords in 2 concentric circles

Figure 9.2: Finding the Rotation Center using its concentric circles

9.4 Homework

1. *Give the minimum angle to carry a pentagon onto itself, using a clockwise rotation around its center ?*
2. *The image of ABC under a translation is* $\triangle A'B'C'$. *Under this translation,* $B(3,-2)$ *maps onto* $B' = (1,-1)$ *and the coordinates of image* A' *are (-2,2). Determine the coordinates of point A.*
3. *A circle A of radius of 3 and a circle B of a radius of 5 have their centers on the same line. Use transformations to explain why circles A and B are similar.*
4. *Determine the coordinates of the figure obtained by rotating A(6, 4) B(1, 3) C(2, 2) D(4, 2) 180 degrees counter-clockwise about the origin.*
5. *Does the transformation* $(x,y) \longrightarrow (2x-1, y-3)$ *represent an isometry?*
6. *The equation of a line* l *is* $2x+3y=5$. *The line* l *is the image of line* m *after a dilation of scale factor 2 with respect to the origin. What is the equation of the original line* m*?*
7. *The following diagram shows the triangles A, B C and D. Describe the rotation that takes:*
   - *A to B, Rotation of   degrees, about (,   ) ;   A to C, Rotation of   degrees, about (,   )*
   - *C to B, Rotation of   degrees, about (,   ) ;   C to D. Rotation of   degrees, about (,   )*

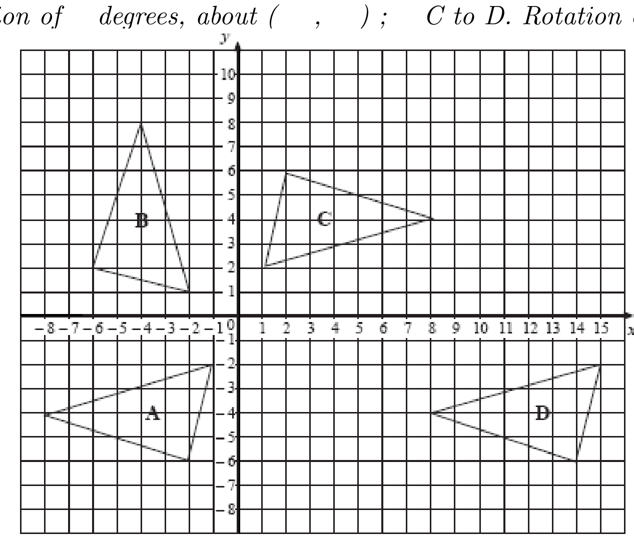


Figure 9.3

8. *A triangle has corners at the points with coordinates (4, 7), (3, 2) and (5, 1).Determine the coordinates of the triangles that are obtained by rotating the original triangle:*
   - *through 90 anticlockwise about (0, 3),*
   - *through 180 about (4, 0),*
   - *through 90 clockwise about (6, 2).*
9. *Can you prove that the equivalencies:*
   - *a 90 degrees counter-clockwise rotation about the origin:* $(x,y) \longrightarrow (-y,x)$.
   - *a 180 degrees counter-clockwise rotation about the origin:* $(x,y) \longrightarrow (-x,-y)$.
   - *a 270 degrees counter-clockwise rotation about the origin:* $(x,y) \longrightarrow (y,-x)$.
10. *Determine the coordinates of the segment obtained by dilating A(1, 5) B(2, 6) by a scale factor of 3 with the center of dilation at D(-3,4).*