AM-GM INEQUALITY

The arithmetic mean-geometric mean inequality, or the AM-GM inequality, is one of the most well-known and powerful inequalities. In words, the AM-GM inequality thate states, the arithmetic mean of two non-negative numbers is at least as large as their geometric mean. Moreover, equality holds precisely when we have a=b.

Thus, the AM-GM inequality is:

$$\boxed{\frac{a+b}{2} \geq \sqrt{ab} \text{ , for the real numbers } a, b \geq 0}$$

Why? Since both sides are positive, we can square then rearrange; this yields

$$(a-b)^2 \geq 0,$$

Note the importance of the "non-negative". Otherwise, if one of the numbers can be negative, and then we could be taking a square root of a negative number. Such an square root is undefined in real numbers, and we would prefer to avoid this situation.

The old question is : does it hold for n? Yes it can be generalized for n non-negative numbers, but the proof is based on mathematical induction, which is a tool that you will study later.

AM-GM for n:

$$\boxed{\frac{a_1+a_2+\cdots+a_{n-1}+a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_{n-1} a_n} \text{ , for the real numbers } a_1, \cdots a_n \geq 0}$$

The equality holds precisely when we have $a_1 = \cdots = a_n$.

Example 1. If the average of two positive real numbers is 3, find the maximum value of their product.

Sol: Using AM-GM we have:

$$\frac{a_1+a_2+a_3+a_4+a_5}{5} \geq \sqrt[5]{a_1 a_2 a_3 a_4 a_5}$$

But $3 = \frac{a_1+a_2+a_3+a_4+a_5}{5}$, so $\sqrt[5]{a_1 a_2 a_3 a_4 a_5} \leq 3$. Thus, $a_1 a_2 a_3 a_4 a_5 \leq 3^5$

Example 2. If $x, y \in \mathbb{R}^+$ and $x+y=8$, then find the minimum value of $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)$.

Sol:The expression can be rewritten as

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) = \frac{1+x+y+xy}{xy} = \frac{9+xy}{xy} = \frac{9}{xy}+1.$$

From AM-GM

$$\frac{x+y}{2} \geq \sqrt{xy} \Leftrightarrow 4 \geq \sqrt{xy} \Leftrightarrow 16 \geq xy \Leftrightarrow \frac{1}{16} \leq \frac{1}{xy} \Leftrightarrow \frac{9}{xy}+1 \geq \frac{25}{16}.$$

Therefore, the minimum value of $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)$ is $\frac{25}{16}$. The equality holds true when $x=y=4$.

Homework

1. For $a, b > 0$ prove that $\frac{a^2}{b^2} + \frac{b^2}{a^2} \geq 2$
2. If a and b are the roots of the equation $x^2 - 5x + 6$, then what is the value of $(\sqrt{a} + \sqrt{b})(\sqrt{a} + \sqrt{b})$?
3. Let x and y two distinct real numbers such that

$$\begin{aligned} x^2 + 3x + 1 &= 0 \\ y^2 + 3y + 1 &= 0 \end{aligned}$$

   Find $x + y$ and $\frac{x}{y} + \frac{y}{x}$.

4. If the perimeter of a rectangle is 17, than what is the greatest possible area?
5. A school wants to fence its rectangular yard of 10,000 square feet. On one side the fence is near a road and the fence costs 2 per foot, while the fence for the other three sides costs 1 per foot. How much of each type of fence will he have to buy in order to keep costs minimal? What is the minimum cost? Hint: Use AM-GM to minimize the cost function

[Optional Exercises for Mathematical Competitions]

6.* Prove that $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 2^2$, for a,b positive real numbers.

7.* Let x_1, x_2, x_3 be the roots of equation $x^3 - x - 1 = 0$. Compute $\frac{2015+x_1}{2015-x_1} + \frac{2015+x_2}{2015-x_2} + \frac{2015+x_3}{2015-x_3}$
Hint: $\frac{2015+x}{2015-x} = \frac{4030}{2015-x} - 1$

8.* Prove that $\frac{x^3+y^3+z^3}{3} \geq xyz$, for x,y, z positive real numbers

9.* Find the minimum value of $\frac{(x-w)(x-y)}{(x-z)(x-t)} + \frac{(x-z)(x-t)}{(x-w)(x-y)}$.

10.* Show that $(x + y)(y + z)(z + x) \geq 8xyz$ for $x, y, z \geq 0$

11.* If x is a positive real number, find the minimum of $x + \frac{1}{x^2}$

12.* Prove that $3x^3 - 6x^2 + \frac{32}{9} \geq 0$. if $x \geq 0$.
Hint: In general if you are trying to use AM-GM, the average of the powers on the LHS should equal the power on the RHS, so try to think on how to split terms.