REDUCING HIGHER ORDER POLYNOMIALS TO QUADRATICS. REMAINDER TH.

A definition defines or explains what a term means. Theorems/Properties/"Facts" must be proven to be true based on postulates and/or already-proven theorems.

Definition 1
A polynomial (function) of degree n:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_p x^p + \cdots + a_2 x^2 + a_1 x + a_0$$

- $a_0,\ a_1,\ \ldots,\ a_n$ *are called coefficients of* P
- $a_p x^p$ *a monomial of degree* p
- $n = deg(P)$

Here we consider that all the coefficients are real numbers.

Example 1

- $P(x) = 7x^6 - 5x^4 + 3x - 11$ has degree 6
- $P(x) = ax + b,\ a \neq 0$ has degree 1
- $P(x) = k,\ k \neq 0$ has degree 0
- $Q(x) = x^3 + x + \dfrac{1}{x}$ is not a polynomial

Property 1
Let P and Q be two polynomials :

- $deg(PQ) = deg(P) + deg(Q)$
- $deg(P+Q) \leq max[\ deg(P)\ ;\ deg(Q)\ ]$

Why? If $P(x) = a_0 + a_1x + ... + a_nx^n$ and $Q = b_0 + b_1x + ... + b_mx_m$, the highest possible degree of PQ is $n+m$, and the coefficient of x^{n+m} in PQ is $a_nb_m = 0$ if and only if $a_n = 0$ or $b_m = 0$. Since $deg(P) = n$ and $deg(Q) = m$ these terms cannot equal 0. Thus, $a_nb_m \neq 0$ and so the degree of PQ is $n + m = deg(P) + deg(Q)$

Definition 2
Two polynomials P *and* Q *are equal, i.e.* $P = Q$ *if :*

- $deg(P) = deg(Q)$
- *the coefficients of the same degree of* P *and* Q *are equal*

Example 2

➔ Prove that $q(x) = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$ are $p(x) = x^4 + 1$ equal:

$$\begin{aligned} q(x) &= (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1) \\ &= x^4 - \sqrt{2}x^3 + x^2 + \sqrt{2}x^3 - 2x^2 + \sqrt{2}x + x^2 - \sqrt{2}x + 1 \\ &= x^4 + 1 \\ q(x) &= p(x) \end{aligned}$$

➔ Identify for what values are $P(x) = 2x^2 - 3x + 4$ and $R(x) = ax^2 + bx + c$ equal:

$a = 2 \quad b = -3 \quad c = 4$ because by definition two polynomials are congruent if all their corresponding coefficients are equal.

Definition 3
We call root of a polynomial (function) p any number x_0 such that $p(x_0) = 0$

Example 3

➔ The real roots of a polynomial are also its x-intercepts. Why?

➔ Can you give an example of a polynomial with no roots in real numbers?

Polynomials can be divided "suspiciously" similar to the real numbers' long division. This is not surprising anymore if we realize that the usual base-10 representation of a number is just a polynomial over 10 instead of the unknown x ($123 = 1 \times 10^2 + 2 \times 10 + 3$). That the division algorithm for polynomials works and gives unique is based on a proof done incrementally on the degree of the polynomials.

Given two polynomials $f(x)$ (the dividend) and $g(x)$ (the divisor), there exist a unique quotient polynomial $q(x)$ and a unique remainder $r(x)$ such that

$$\boxed{f(x) = q(x)g(x) + r(x) \quad \text{and } 0 \leq \deg(r) < \deg(g)}$$

How is it done? By multiplying in order to reduce at each step the degree of $f(x) = a_n x^n + \cdots$. Formally, we can say that we multiply the divisor $g(x) = b_m x^m + \cdots$ by $\frac{a_n}{b_m} x^{n-m}$, but let us do an example.

Example 4
Divide $f(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$ by $(x - 1)$ similarly to the numbers division.

$$\begin{array}{rrrrrrrrr|l} x^4 & - & 7x^3 & + & 17x^2 & - & 17x & + & 6 & x-1 \\ \cline{10-10} x^4 & - & x^3 & & & & & & & x^3 - 6x^2 + 11x - 6 \\ \hline & - & 6x^3 & + & 17x^2 & - & 17x & + & 6 & \\ & - & 6x^3 & + & 6x^2 & & & & & \\ \cline{2-9} & & & + & 11x^2 & - & 17x & + & 6 & \\ & & & + & 11x^2 & - & 11x & & & \\ \cline{4-9} & & & & & - & 6x & + & 6 & \\ & & & & & - & 6x & + & 6 & \\ \cline{6-9} & & & & & & & & 0 & \end{array}$$

Conclusion : $\boxed{f(x) = (x - 1)(3x^3 - 4x^2 + 5x + 6)}$

Theorem 1
(Remainder Th.) Let P a polynomial of degree n, then he remainder of the division of P to $(x - x_0)$ is $P(x_0)$.

Proof Idea 1
Why ? We saw it in the example, but for a general proof we use the division identity for polynomials : $P(x) = (x - x_0)Q(x) + R(x)$, $0 \leq deg(R(x)) < 1 = deg(x - x_0)$ so $deg(R(x)) = 0$ so $R(x) = R$ is a number. The expression $P(x) = (x - x_0)Q(x) + R$ holds for any x, hence for $x = x_0$. If we evaluate it in x_0 we obtain $P(x_0) = 0Q(x) + R = R$

Consequence 1
When you are faced with a higher degree polynomial and you only know how to solve quadratics a fast way to search for a simple "elementary" root is to use the Remainder Theorem :
try and see if replacing x with 1, -1, $0, \dots$ leads to $P(1) = 0$, or $P(-1) = 0$, or $P(0) = 0, \cdots$.

Example 5
What is the degree of the remainder of the division of $f(x) = 3x^4 - x^3 + x^2 + 11x + 6$ by $(x+1)$? What about $(x+2)$ What is the degree of the new remainder? Can you find them without performing the division? Factorize $f(x)$

$f(-1) = 0$ so using the Remainder Theorem $x + 1$ is a factor of $f(x)$
$f(-2) \neq 0$ so using the Remainder Theorem$x + 2$ is not a factor of $f(x)$

$$\begin{aligned} f(x) &= (x+1)g(x) \\ &= (x+1)(ax^3 + bx^2 + cx + d) \\ &= ax^4 + bx^3 + cx^2 + dx + ax^3 + bx^2 + cx + d \\ &= ax^4 + (b+a)x^3 + (c+b)x^2 + (d+c)x + d \end{aligned}$$

The polynomials $3x^4 - x^3 + x^2 + 11x + 6$ and $ax^4 + (b+a)x^3 + (c+b)x^2 + (d+c)x + d$ are equal, so their corresponding coefficients are identical $\left\{\begin{array}{ccc} a &=& 3 \\ b+a &=& -1 \\ c+b &=& 1 \\ d+c &=& 11 \\ d &=& 6 \end{array}\right.$ so : $\left\{\begin{array}{ccc} a &=& 3 \\ b &=& -4 \\ c &=& 5 \\ d &=& 6 \end{array}\right.$

Conclusion : $\boxed{f(x) = (x+1)(3x^3 - 4x^2 + 5x + 6)}$

Example 6
Factorise $P(x) = x^3 - 7x + 6$

1. Look for "elementary" roots $(\pm 2, \pm 1, 0)$ and factorize P(x)
2. Find the remainder of division of P(x) to x. Is it equal to $P(0)$?
3. Find the remainder of division of P(x) to $x + 1$. Is it equal to $P(-1)$?

➔ $P(2) = 0$, $P(1) = 0$, so $(x-2)$, $(x-1)$ are factors

➔ $P(x) = (x-2)(x-1)Q(x)$ so $deg(P) = deg(x-2) + deg(x-1) + deg(Q)$
so, $deg(Q) = 3 - 1 - 1 = 1$

$$\begin{aligned} P(x) &= (x-2)(x-1)(ax+b) = (x^2 - x - 2x + 2)(ax+b) \\ &= (x^2 - 3x + 2)(ax+b) = ax^3 + bx^2 - 3ax^2 - 3bx + 2ax + 2b \\ &= ax^3 + (b-3a)x^2 + (-3b+2a)x + 2b \\ P(x) &= x^3 - 7x + 6 \end{aligned}$$

$$\left\{\begin{array}{ccc} a &=& 1 \\ b - 3a &=& 0 \\ -3b + 2a &=& -7 \\ 2b &=& 6 \end{array}\right. \quad \text{donc :} \quad \left\{\begin{array}{ccc} a &=& 1 \\ b &=& 3 \end{array}\right.$$

Conclusion : $\boxed{P(x) = (x-2)(x-1)(x+3)}$

Example 7
When $x^2 - 5x + 4$ is divided by $(x - m)$ the remainder is -2. Find all possible values of m.

Let $f(x) = x^2 - 5x + 4$, then $P(m) = m^2 - 5m + 4 = -2 \Leftrightarrow m^2 - 5m + 6 = 0 \Leftrightarrow (m-3)(m-2) = 0$

Theorem 2
A quadratic has at most 2 real roots. A polynomial P of degree n with real coefficients has at most n real roots.

Proof Idea 2
Why ? Think of the degree of $(x-a)(x-b)$. What about the degree of $(x-a_1)...(x-a_n)$

> **Definition 4**
> *A rational expression is a fraction in which the numerator and/or the denominator are polynomials.*

Equations that contain rational expressions are called rational equations. One method for solving rational equations is to rewrite the rational expressions in terms of a common denominator.

Homework

1. Rationalize $\frac{6}{\sqrt{192}+2\sqrt{108}+8\sqrt{27}}$
2. What are the last digits of the numbers $M = 5^{555} + 6^{666}$ and $N = 5^{555} + 6^{666} + 7^{777}$
3. Solve for x: $(x-51)^7 = \frac{1}{128}$.
4. Find all real values x that satisfy $\frac{\sqrt{7+\sqrt[7]{x}}}{\sqrt{7+\sqrt[7]{x}}} = \sqrt{7}$.
5. Factorize $x^4 + x^2 + 1$
6. Factorize. Look for special identities:difference of squares, sum/difference of cubes, the square of a binomial.
   (a) $(3x+1)^2 - 49$
   (b) $(4x-y)^2 - 64y^2$
   (c) $3x^3 + 24y^3$
   (d) $(y-a)^2 - y + a$
   (e) $4x^2y + 12xy^2 + 9y^3$
   (f) $x^2 + 18x + 81$
   (g) $x^2 - 22x + 121$
   (h) $64x^2 + 16x + 1$
   (i) $25x^2 + 40x + 16$
   (j) $x^2 - 2xy + 5x - 10y$
   (k) $x^2 + 3x - 3y - y^2$
   (l) $a^2b^2 + a^4b^2 + 2a^2c + 2a^4c + b^2 + 2c$
   (m) $a^3b^6 + 8$
7. If $25^x - 4^x = 14$ and $5^x + 2^x = 7$, compute $5^x - 2^x$.
8. The product of four consecutive positive integers is 1 less than 1805^2. What is the least of these four numbers?
9. Let n, m integers $-10 \leq n \leq -4$ and $1 < m \leq 7$, what is the least possible value of $\left(\frac{1}{n}+\frac{1}{m}\right)\left(\frac{1}{n}-\frac{1}{m}\right)$?
10. Solve
    (a) $\frac{7x}{x-2} - \frac{5}{3-x} = 0.$
    (b) $\frac{x^2+5x+4}{1+\frac{x^2+x-1}{x+5}} = 1.$
    (c) $\frac{5x}{2x+1} + 3 = 4$
    (d) $\frac{3x+6}{x^2-4} = 0$
11. Use polynomial division to simplify $\frac{27x+9x-3x-10}{3x-2}$and $\frac{3x^4-x^3+8x^2+5x+3}{x^2-x+3}$
12. For $q(x) = 2x^3 - x + 3$ evaluate $\frac{q(x+3)-q(3)}{x}$.
13. What is the remainder when $x^{91} + 91x + 91$ is divided by $x+1$? (Do not divide!)
14. When $x^4 + 2x^3 + 3x + k$ is divided by $(x+3)$ the remainder is 23. Find the value of k.
15. If $x^3 - x^2 + mx + 6$ has a remainder of 4 when divided by$(x+2)$, find the value of m.
16.* Let $f(x)$ be a quadratic polynomial in x with real coefficients such that $f(x)f(2x^2) = f(2x^3+x)$. Find the coefficients of $f(x)$.
17.** Find x real such that $x + \sqrt{(x+1)(x+2)} + \sqrt{(x+2)(x+3)} + \sqrt{(x+3)(x+1)} = 4$.