PÓLYA-REDFIELD METHOD

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Based on the book *Combinatorics* by Daniel Marcus.

Given a string $x_1x_2x_3...x_n$, a rotation of the string has the form $x_{i+1}x_{i+2}x_{i+3}...x_nx_1x_2...x_i$. Given a string $x_1x_2...x_n$, the set of all strings obtained by rotating the string are called an *orbit* since rotating n times returns to the original string.

Problem 1. Consider strings of length 4 on letters A and B. Find all of the orbits.

Problem 2. Find the orbits of strings $AAABBB$, $AABAAB$ and $ABABAB$. For each string, how many rotations leave the string in its original position? This set of rotations is called the *stabilizer*.

Problem 3. Let R_i denote rotation by i places,

$$R_i(x_1x_2...x_n) = x_{i+1}x_{i+2}...x_nx_1x_2...x_i. \tag{1}$$

Prove that $R_iR_j = R_{i+j \bmod n}$.

Problem 4. Given a string of length n, prove that the size of its orbit times the size of its stabilizer is equal to n.

Problem 5. How many ways are there to color the edges of an equilateral triangle with three colors? Colors may be repeated. Two colorings are considered equal if one is obtained from the other by rotating the triangle.

Problem 6. Consider the sum

$$\begin{aligned}&AAAA + AAAB + AABA + AABB + ABAA + ABAB + ABBA + ABBB\\&+ BAAA + BAAB + BABA + BABB + BBAA + BBAB + BBBA + BBBB.\end{aligned}$$

Replace each string with $1/o$ where o is the length of the orbit. Show that this sum gives the number of orbits.

Problem 7. Prove that

$$\begin{aligned}&(R_1 + R_2 + R_3 + R_4)(AAAA + AAAB + ... + BBBB)\\&= 4(AAAA + AAAB + ... + BBBB).\end{aligned}$$

Problem 8. Consider all strings of length n on m letters. Prove that the sum of the size of the stabilizers is equal to n times the number of orbits under rotation.

Problem 9. The invariant number r_i of rotation R_i is the number of strings fixed by the rotation. Prove

$$\sum_{i=1}^{n} r_i = n\#\{\text{Orbits}\}.$$

Problem 10. Find the number of orbits on 12 letter strings that use the letters A, B.

Problem 11. Find all colorings of a 2×2 square with 3 colors if the square can be both rotated and reflected.

Problem 12. How many ways can five beads be arranged on a circular bracelet if each bead can be one of three colors. The bracelet can be rotated or flipped.

Problem 13. A 3×3 square is colored with the two colors A and B, and is allowed to be rotated but not flipped. Find the invariant numbers of the rotations by 0, 90, 180 and 270 degrees and calculate the number of colorings.

Explain why the coefficient on $A^x B^{9-x}$ in the *pattern inventory*

$$\frac{(A+B)^9 + 2(A+B)(A^4+B^4)^2 + (A+B)(A^2+B^2)^4}{4} \tag{2}$$

gives the number of colorings with x A's and $9-x$ B's.

If the positions in the square are numbered

1	2	3
4	5	6
7	8	9

a 90 degree counterclockwise rotation maps 1 to 7, 7 to 9, 9 to 3 and 3 to 1, 2 to 4, 4 to 8, 8 to 6 and 6 to 2, and 5 to 5. This can be represented using a *cycle code* as $(1,7,9,3)(2,4,8,6)(5)$. The 180 degree rotation is represented by $(1,9)(7,3)(2,8)(4,6)(5)$. The first map is represented by $X_1 X_4^2$ and the second by $X_1 X_2^4$, since the first has one cycle of length 1 and 2 cycles of length 4, while the second has one cycle of length 1 and four cycles of length 2. The *cycle index polynomial* of the set of rotations is the average of the cycle codes,

$$\frac{X_1^4 + 2X_1X_4^2 + X_1X_2^4}{4}.$$

Problem 14. Prove that the *pattern inventory* on k colors is the result of substituting $X_i = A_1^i + \dots + A_k^i$ into the cycle index polynomial.

Problem 15. There are 24 different orientations of a cube. Find the number of ways of coloring the cube using k colors.

Problem 16. The are 12 different orientations of a regular tetrahedron. Find the number of ways of coloring the faces using k colors.