MATH CLUB: AMC PRACTICE PROBLEMS

FEB 11, 2018

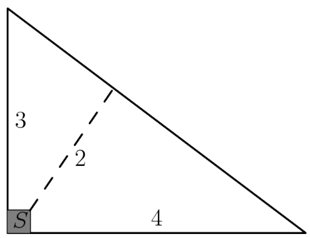
Today we did some practice problems preparing for AMC 10/12B.

Most problems below are from this year AMC 10/12A, given last week. You can find full list of problems with solutions here:

`https://artofproblemsolving.com/wiki/index.php?title=2018_AMC_10A_Problems`
`https://artofproblemsolving.com/wiki/index.php?title=2018_AMC_12A_Problems`

1. A number m is randomly selected from the set $\{11, 13, 15, 17, 19\}$, and a number n is randomly selected from $\{1999, 2000, 2001, \dots, 2018\}$. What is the probability that m^n has a units digit of 1?

2. Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



3. Let A be the set of positive integers that have no prime factors other than 2, 3, or 5. Find the infinite sum of the reciprocals of the elements of A:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{12}+\frac{1}{15}+\frac{1}{16}+\frac{1}{18}+\frac{1}{20}+\cdots$$

4. Triangle ABC is an isosceles right triangle with $AB = AC = 3$. Let M be the midpoint of hypotenuse $\overline{BC}$. Points I and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle EMI has area 2, find the length CI.

5. Which of the following polynomials has the greatest real root?

(A) $x^{19} + 2018x^{11} + 1$ **(B)** $x^{17} + 2018x^{11} + 1$ **(C)** $x^{19} + 2018x^{13} + 1$
(D) $x^{17} + 2018x^{13} + 1$ **(E)** $2019x + 2018$

6. Alice, Bob, and Carol play a game in which each of them chooses a real number between 0 and 1. The winner of the game is the one whose number is between the numbers chosen by the other two players. Alice announces that she will choose her number uniformly at random from all the numbers between 0 and 1, and Bob announces that he will choose his number uniformly at random from all the numbers between $\frac{1}{2}$ and $\frac{2}{3}$. Armed with this information, what number should Carol choose to maximize her chance of winning?

Turn over for hints

Hints

2. There are many ways to do this problem. One is using coordinates and the formula for distance from a point to a line. Another way is dividing the field into triangles (except for S) and use the fact that their areas must add up to the area of the whole triangle.

3.

$$\sum_{k,l,m\geq 0} \frac{1}{2^k 3^l 5^m} = \Big(\sum_{k\geq 0} \frac{1}{2^k}\Big)\Big(\sum_{l\geq 0} \frac{1}{3^l}\Big)\Big(\sum_{m\geq 0} \frac{1}{5^m}\Big)$$

4. Prove that $MI = ME$. One way to do it is by using 90° rotation around M.

5. First, note that each of these roots must be a negative number very close to zero. Second, for a small positive number ε, we have $\varepsilon^{19} < \varepsilon^{17}$, and thus $(-\varepsilon)^{19} > (-\varepsilon)^{17}$.

6. For any c between $\frac{1}{2}$ and $\frac{2}{3}$, probability that a number a which is randomly chosen on $[0,1]$ satisfies $a < c$ is equal to c, and probability that a number b randomly chosen on $[\frac{1}{2}, \frac{2}{3}$ satisfies $c < b$ is equal to $(\frac{2}{3} - c)/(\frac{2}{3} - \frac{1}{2})$.